

Hindman's Sums and Products Conjecture

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In September 2026, OpenAI released a preprint (OpenAI 2026) announcing a proof of a conjecture formulated by Hindman in 1979. The conjecture concerns a beautiful problem in Ramsey theory: if we colour all positive integers using finitely many colours, can we always find arbitrarily many integers such that all their sums and products are of the same colour?

To understand the problem, let us recall some classical results. In 1916, Schur (Schur 1916) proved that, for every colouring of the positive integers using finitely many colours, there exist positive integers x, y, z of the same colour such that

$$x + y = z.$$

A set of integers is called *monochromatic* if all its elements have the same colour. Thus, Schur's theorem says that every finite colouring of the positive integers contains a monochromatic set of the form

$$\{x, y, x + y\}.$$

In 1974, Hindman (Hindman 1974) proved a remarkable generalization. For every finite colouring of the positive integers, there exists an infinite sequence

$$a_1, a_2, a_3, \dots$$

such that all sums of finitely many distinct terms of this sequence have the same colour. For example, the numbers

$$a_1, a_2, a_3, a_1 + a_2, a_1 + a_3, a_2 + a_3, a_1 + a_2 + a_3$$

must all have the same colour, and the same is true for all sums involving any finite number of terms.

In a previous post, we discussed Hindman's finite sums theorem and its recent density generalizations.

There is also a multiplicative version of Hindman's theorem. In every finite colouring of the positive integers, we can find an infinite sequence $a_1, a_2, \dots$ such that all products of finitely many distinct terms have the same colour.

Indeed, this follows easily from Hindman's theorem by considering the induced colouring of the powers of 2: if all the sums

$$n_{i_1} + \cdots + n_{i_k}$$

have the same induced colour, then all the products

$$2^{n_{i_1}} \cdots 2^{n_{i_k}}$$

have the same original colour.

But can we combine these two statements? In other words, can we find a set of integers for which *both* all finite sums and all finite products have the same colour?

This turns out to be much harder.

For a finite set $A = \{a_1, \dots, a_m\}$ of distinct positive integers, define its set of *finite sums* by

$$FS(A) = \left\{ \sum_{i \in I} a_i : \emptyset \neq I \subseteq \{1, \dots, m\} \right\},$$

and its set of *finite products* by

$$FP(A) = \left\{ \prod_{i \in I} a_i : \emptyset \neq I \subseteq \{1, \dots, m\} \right\}.$$

In both definitions, each element is used at most once, and subsets consisting of a single element are allowed.

The problem is to find A such that

$$FS(A) \cup FP(A)$$

is monochromatic.

For $m = 2$, this asks whether every finite colouring contains a monochromatic set

$$\{x, y, x + y, xy\}.$$

This question was formulated by Hindman (Hindman 1979) in 1979 and attracted considerable attention.

The case of two colours was settled by computer-assisted arguments of Graham and Hindman. However, even the case of three colours remained open for decades.

In 2017, Moreira (Moreira 2017) proved that every finite colouring of the positive integers contains a monochromatic set of the form

$$\{x, x + y, xy\}.$$

This was a major breakthrough, but it did not guarantee that y itself had the same colour.

Moreira's proof used methods from topological dynamics, a branch of mathematics studying how points evolve under repeated transformations.

Subsequently, Bowen and Sabok (Bowen and Sabok 2024) proved in 2024 that every finite colouring of the *rational numbers* contains a monochromatic set $\{x, y, x + y, xy\}$. An even stronger result over the rational numbers was obtained by Alweiss (Alweiss 2023), who proved that for every positive integer m , every finite colouring of the rational numbers contains m distinct elements whose finite sums and finite products all have the same colour.

However, this did not resolve Hindman's original conjecture about *positive integers*. One cannot simply multiply all the rational numbers by a common denominator: their sums would be multiplied by that denominator, but products of two or more numbers would be multiplied by different powers of it. An arbitrary colouring need not preserve colours under such transformations.

The new OpenAI preprint (OpenAI 2026) claims to settle Hindman's conjecture for positive integers, in full generality.

Theorem 1 (OpenAI) *For every finite colouring of the positive integers and every positive integer m , there exist positive integers*

$$a_1 < a_2 < \cdots < a_m$$

such that all the numbers in

$$\left\{ \sum_{i \in I} a_i : \emptyset \neq I \subseteq \{1, \dots, m\} \right\} \cup \left\{ \prod_{i \in I} a_i : \emptyset \neq I \subseteq \{1, \dots, m\} \right\}$$

have the same colour.

For $m = 2$, we obtain the original conjecture concerning $\{x, y, x + y, xy\}$.

For $m = 3$, the theorem guarantees the existence of positive integers $a < b < c$ such that all eleven numbers

$$a, b, c, a + b, a + c, b + c, a + b + c, \\ ab, ac, bc, abc$$

have the same colour.

In fact, the preprint proves that the elements $a_1, \dots, a_m$ can be chosen very far apart. Consequently, all $2^m - 1$ nonempty subset sums can be made distinct, and the same holds for the $2^m - 1$ nonempty subset products. Moreover, the two sets can be chosen to have only the original m elements in common.

Thus, the theorem guarantees monochromatic sets containing

$$2(2^m - 1) - m$$

distinct integers, for every prescribed m .

Notice that the theorem does not assert the existence of an *infinite* set A whose finite sums and products are all monochromatic. In fact, Hindman (Hindman 1980) proved in 1980 that this stronger statement is false. There exists a finite colouring of the positive integers for which no infinite set has all its pairwise sums and pairwise products of the same colour.

Therefore, the restriction to arbitrarily large *finite* sets is essential.

Let us briefly discuss the proof. The authors combine ideas from classical Ramsey theory, analytic number theory, and the theory of structured arithmetic sequences.

The first step is to construct groups of integers whose products have a common colour. This part uses the classical finite Ramsey theorem and Hindman's finite sums theorem. The main challenge is then to ensure that all the required sums have that same colour.

To achieve this, the authors develop two ingredients, called the *Prediction Principle* and the *Alignment Principle*. The first replaces certain complicated colour patterns by more regular functions, whose behaviour under addition can be studied. The second makes it possible to choose the integers so that the desired colour is preserved when the required sums are formed.

The proof also uses carefully chosen divisibility conditions and weighted counting arguments to ensure that the abstract configurations produced by these methods correspond to actual positive integers.

This is a highly nontrivial combination of additive and multiplicative techniques. Although addition and multiplication are individually well understood in Ramsey theory, controlling them simultaneously has proved exceptionally difficult.

The theorem also has an interesting connection with the subject of another earlier post, in which we discussed the sizes of the sets of simple sums and products of integers. That problem asks how large the sets $FS(A)$ and $FP(A)$ must be. Hindman's conjecture asks a completely different question: can we force *both* sets to lie in one colour class, regardless of the colouring?

The OpenAI result is currently presented as a preprint, and independent verification of its proof remains important. If confirmed, it will resolve a problem that has been open for almost half a century and provide a striking illustration of the power of Ramsey theory: no matter how we colour the positive integers with finitely many colours, we cannot prevent arbitrarily large finite sets from having all their subset sums and subset products in one colour.

References

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