

Prime Factors of Consecutive Integers

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On September 24, 2026, OpenAI released a preprint (OpenAI 2026) announcing a proof of a conjecture formulated by Erdős and Pomerance in 1978. The conjecture concerns the largest prime factors of two consecutive integers and predicts that their sizes behave independently in a certain statistical sense. The paper also claims to prove that, among two consecutive integers, each has the larger prime factor approximately half the time.

In a recent post, we discussed the two-point Chowla conjecture, which predicts that the parities of the numbers of prime factors of two consecutive integers behave independently. The new result concerns a different characteristic of prime factorization: the size of the *largest* prime factor.

For an integer $n \geq 2$, let $P^+(n)$ denote its largest prime factor. For example,

$$84 = 2^2 \cdot 3 \cdot 7, \quad 85 = 5 \cdot 17,$$

so

$$P^+(84) = 7, \quad P^+(85) = 17.$$

We also put $P^+(1) = 1$.

Since consecutive integers are relatively prime, meaning that they have no common prime factors, their prime factorizations are necessarily different. However, this does not imply that the *sizes* of their largest prime factors behave independently.

For example, suppose that n has no prime factor greater than $\sqrt{n}$. Does this make it more or less likely that $n + 1$ also has no prime factor greater than $\sqrt{n}$? Or does knowing the size of $P^+(n)$ give essentially no information about the size of $P^+(n + 1)$?

To formulate this question precisely, recall that an integer is called *y-smooth* if all its prime factors are at most y . For example, $72 = 2^3 \cdot 3^2$ is 3-smooth, while $70 = 2 \cdot 5 \cdot 7$ is not.

The distribution of smooth numbers has been studied extensively since the work of Dickman in 1930 (Dickman 1930). In particular, for every fixed real number $a \in (0, 1)$, the proportion of positive integers $n \leq X$ satisfying

$$P^+(n) \leq n^a$$

approaches a limit as $X \rightarrow \infty$. Let us denote this limit by $D(a)$:

$$D(a) = \lim_{X \rightarrow \infty} \frac{\#\{2 \leq n \leq X : P^+(n) \leq n^a\}}{X}.$$

Here $\#A$ denotes the number of elements of a finite set A .

The function $D(a)$ is related to the classical *Dickman function*, which describes the distribution of smooth numbers. For example, it is known that

$$D\left(\frac{1}{2}\right) = 1 - \log 2 \approx 0.30685,$$

where $\log$ denotes the natural logarithm. Thus, approximately 30.7% of positive integers have no prime factor greater than their square root.

In 1978, Erdős and Pomerance (Erdős and Pomerance 1978) conjectured that the corresponding events for n and $n + 1$ are asymptotically independent. More precisely, they predicted that, for every fixed $a, b \in (0, 1)$, the proportion of integers n satisfying both

$$P^+(n) \leq n^a \quad \text{and} \quad P^+(n + 1) \leq n^b$$

should approach $D(a)D(b)$, the product of the individual limiting proportions.

The conjecture is natural, but proving it turned out to be very difficult. In their original paper, Erdős and Pomerance established that each of the inequalities

$$P^+(n) < P^+(n + 1) \quad \text{and} \quad P^+(n) > P^+(n + 1)$$

holds for a positive proportion of all positive integers. However, they could not determine these proportions precisely.

In 2018, Teräväinen (Teräväinen 2018) established the predicted independence in a weaker, *logarithmically averaged* sense, where each integer n receives weight $1/n$ instead of being counted equally. Subsequently, Tao and Teräväinen (Tao and Teräväinen 2019) obtained the ordinary independence statement outside an exceptional set of scales. However, the full conjecture, involving ordinary proportions for all sufficiently large cutoffs, remained open.

The new OpenAI preprint (OpenAI 2026) claims to establish exactly this.

Theorem 1 (OpenAI) For every $a, b \in (0, 1)$,

$$\lim_{X \rightarrow \infty} \frac{\#\{2 \leq n \leq X : P^+(n) \leq n^a, P^+(n+1) \leq n^b\}}{X} = D(a)D(b),$$

where

$$D(a) = \lim_{X \rightarrow \infty} \frac{\#\{2 \leq n \leq X : P^+(n) \leq n^a\}}{X}.$$

Moreover,

$$\lim_{X \rightarrow \infty} \frac{\#\{2 \leq n \leq X : P^+(n) < P^+(n+1)\}}{X} = \frac{1}{2}.$$

For example, taking $a = b = 1/2$, the theorem gives

$$\begin{aligned} \lim_{X \rightarrow \infty} \frac{1}{X} \#\{2 \leq n \leq X : P^+(n) \leq \sqrt{n}, \\ P^+(n+1) \leq \sqrt{n}\} \\ = (1 - \log 2)^2 \approx 0.09416. \end{aligned}$$

Thus, approximately 9.4% of positive integers n have the property that both n and $n+1$ have no prime factor greater than $\sqrt{n}$.

The second assertion of the theorem is particularly striking. It states that

$$P^+(n) < P^+(n+1)$$

holds for asymptotically half of all positive integers n .

Why does this follow from the independence statement? The theorem implies that the two quantities

$$\frac{\log P^+(n)}{\log n} \quad \text{and} \quad \frac{\log P^+(n+1)}{\log n}$$

have the same limiting distribution and become statistically independent.

Moreover, this limiting distribution is continuous, so the probability that the two limiting quantities are equal is zero. By symmetry, either one is larger with probability $1/2$.

Notice that this does not mean that the prime factors of n and $n+1$ are literally independent. They cannot share any prime factors at all! The theorem asserts independence only for the logarithmic sizes of their largest prime factors, in the limit as n becomes large.

Let us briefly discuss the proof. The authors first divide prime numbers into several ranges according to their sizes and study how many prime factors of n belong to each range. This reduces the problem of estimating the largest prime factor to a collection of more manageable counting problems.

The main difficulty is to show that the corresponding counts for n and $n + 1$ become independent. The authors develop a method involving carefully weighted factorizations of integers, which transforms the original problem into estimates over short intervals. They then combine these estimates with results about the statistical behavior of multiplicative functions, building on earlier work of Matomäki and Radziwiłł (Matomäki and Radziwiłł 2016). This eventually yields the required asymptotic independence.

The result is particularly interesting because the two integers under consideration are consecutive and therefore strongly related arithmetically. Nevertheless, when we measure the sizes of their largest prime factors on a logarithmic scale, this relationship disappears in the limiting statistical distribution.

The OpenAI repository also reports a formalization of the main result in Lean, although independent mathematical scrutiny of the preprint remains important.

If confirmed, the theorem will resolve a conjecture that has remained open for almost fifty years. Together with the recent progress on the two-point Chowla conjecture, it provides another remarkable illustration of the apparent randomness hidden within the prime factorizations of consecutive integers.

References

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