

Goldfeld's Conjecture Is Proved

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In October 2026, OpenAI released two preprints (OpenAI 2026a, 2026b) claiming to resolve a famous conjecture formulated by Dorian Goldfeld in 1979. The conjecture concerns the distribution of ranks of elliptic curves and predicts a remarkable phenomenon: in certain natural families of elliptic curves, exactly half should have analytic rank 0, and the other half should have analytic rank 1. Moreover, their average analytic rank should approach $1/2$.

In a previous post, I discussed Goldfeld's conjecture in connection with the classical *congruent number problem* and a recent theorem of Burungale and Tian (Burungale and Tian 2026). Let me briefly recall the necessary background.

An *elliptic curve* over the rational numbers is a curve given by an equation

$$E: y^2 = x^3 + ax + b,$$

where $a, b \in \mathbb{Q}$ satisfy

$$4a^3 + 27b^2 \neq 0.$$

The last condition ensures that the curve has no singularities.

The rational solutions (x, y) of this equation, together with one additional point called the *point at infinity*, form a group under a natural addition operation. A classical theorem of Mordell and Weil states that this group is finitely generated. In particular, it is isomorphic to

$$\mathbb{Z}^r \oplus T,$$

where $r \geq 0$ is an integer and T is a finite group. The integer r is called the *rank* of the elliptic curve. If $r = 0$, the curve has only finitely many rational points; if $r > 0$, it has infinitely many.

There is another important quantity associated with an elliptic curve, called its *analytic rank*. To define it, mathematicians construct a complex function $L(E, s)$, called the *L-function* of E . This function encodes information about the number of solutions of the equation defining E modulo different prime numbers.

The function $L(E, s)$ is initially defined by an infinite product, but it has a continuation to the entire complex plane. Its *analytic rank*, denoted by $r_{\text{an}}(E)$, is the order of vanishing of $L(E, s)$ at $s = 1$. In other words, $r_{\text{an}}(E) = r$ if

$$L(E, s) = c(s - 1)^r + \text{higher powers of } (s - 1),$$

where $c \neq 0$. In particular,

$$r_{\text{an}}(E) = 0 \iff L(E, 1) \neq 0,$$

while $r_{\text{an}}(E) = 1$ means that $L(E, 1) = 0$ but the derivative $L'(E, 1)$ is nonzero.

The famous *Birch and Swinnerton-Dyer conjecture* predicts, among other things, that the analytic rank is always equal to the rank of the group of rational points. This remains one of the great open problems in number theory.

Now fix an elliptic curve

$$E : y^2 = x^3 + ax + b.$$

For any nonzero square-free integer d , its *quadratic twist* by d is the elliptic curve

$$E^{(d)} : y^2 = x^3 + ad^2x + bd^3.$$

Recall that an integer is called square-free if it is not divisible by the square of any prime. We allow both positive and negative square-free integers.

The curves E and $E^{(d)}$ become isomorphic if we allow coordinates involving $\sqrt{d}$, but over the rational numbers they may have very different properties. In particular, their ranks can be different.

In 1979, Goldfeld (Goldfeld 1979) conjectured that, as we vary d , the analytic ranks of $E^{(d)}$ should almost always be either 0 or 1, and these two possibilities should occur equally often.

More precisely, let

$$\mathcal{D}(X) = \{d \in \mathbb{Z} : 0 < |d| \leq X, |d| \text{ is square-free}\}.$$

Then Goldfeld's conjecture predicts that, for every fixed elliptic curve E , the proportion of integers $d \in \mathcal{D}(X)$ for which $E^{(d)}$ has analytic rank 0 should tend to $1/2$ as $X \rightarrow \infty$. The same should hold for analytic rank 1.

The conjecture also predicts that the *mean analytic rank*

$$\frac{1}{|\mathcal{D}(X)|} \sum_{d \in \mathcal{D}(X)} r_{\text{an}}(E^{(d)})$$

should tend to $1/2$.

Notice that the mean assertion is stronger than the two density assertions. Even if almost all curves have analytic rank 0 or 1, a small exceptional collection of curves with very large ranks could make a significant contribution to the average.

The conjecture has attracted considerable attention because it offers a statistical description of the arithmetic complexity of elliptic curves. Rather than trying to calculate the rank of one particular curve, we ask how the ranks are distributed in an infinite family.

There has been substantial progress towards Goldfeld's conjecture. In particular, Alexander Smith proved important results concerning the ranks of the elliptic curves

$$E_n : y^2 = x^3 - n^2x,$$

which are associated with the congruent number problem. Subsequently, Burungale and Tian (Burungale and Tian 2026) proved that almost all positive square-free integers $n \equiv 1, 2, 3 \pmod{8}$ give curves E_n of analytic rank 0, establishing the rank-zero density assertion for this special family. We discussed these results in the earlier post.

In 2025, Smith (Smith 2025) obtained a general distribution theorem for an important algebraic invariant of quadratic twists, known as the 2^∞ -Selmer corank. This invariant arises from studying the solvability of certain auxiliary equations associated with elliptic curves. It is always at least the rank of the curve.

Smith proved that, for quadratic twists of every elliptic curve over $\mathbb{Q}$, this invariant equals 0 and 1 with density 1/2 each. If the Birch and Swinnerton-Dyer conjecture were known, this would imply the corresponding analytic-rank statement. However, obtaining that conclusion unconditionally remained a major challenge.

The two new OpenAI preprints (OpenAI 2026a, 2026b) claim to overcome this difficulty and prove Goldfeld's conjecture in full.

Theorem 1 (OpenAI) *Let E be any elliptic curve over $\mathbb{Q}$, and let*

$$\mathcal{D}(X) = \{d \in \mathbb{Z} : 0 < |d| \leq X, |d| \text{ is square-free}\}.$$

Then, for $j = 0, 1$,

$$\lim_{X \rightarrow \infty} \frac{\#\{d \in \mathcal{D}(X) : r_{\text{an}}(E^{(d)}) = j\}}{|\mathcal{D}(X)|} = \frac{1}{2}.$$

Moreover,

$$\lim_{X \rightarrow \infty} \frac{1}{|\mathcal{D}(X)|} \sum_{d \in \mathcal{D}(X)} r_{\text{an}}(E^{(d)}) = \frac{1}{2}.$$

Thus, if we consider more and more quadratic twists of any fixed elliptic curve, approximately half have analytic rank 0, and half have analytic rank 1. Curves of analytic rank 2 or higher become increasingly rare: their proportion tends to zero.

The theorem is completely general. It applies to every elliptic curve over $\mathbb{Q}$, without restrictions on its coefficients or other arithmetic properties. Both positive and negative twisting parameters are included.

Let us briefly discuss the proof. The first preprint (OpenAI 2026a) builds on Smith's theorem concerning the distribution of Selmer groups. Its main new ingredient is a result showing that, whenever the 2^∞ -Selmer corank of an elliptic curve is 0 or 1, its analytic rank must be equal to that corank. The proof combines properties of rational points on elliptic curves with sophisticated methods for studying their L -functions. Together with Smith's distribution theorem, this gives the two density assertions.

The second preprint (OpenAI 2026b) addresses the additional difficulty of controlling the contribution of rare curves with large analytic ranks. It establishes new estimates showing that this contribution becomes negligible as $X \rightarrow \infty$, completing the proof of the mean assertion.

The density result has another important consequence. For almost all quadratic twists of every fixed elliptic curve over $\mathbb{Q}$, the analytic rank is equal to the rank of the group of rational points. Thus, the preprints also claim substantial progress towards the Birch and Swinnerton-Dyer conjecture, although they do not establish its full statement for every elliptic curve.

If these proofs withstand independent scrutiny, they will resolve a famous conjecture that has been open for almost half a century. The conclusion is remarkably simple: among the quadratic twists of any fixed elliptic curve, analytic ranks 0 and 1 each occur half the time, and even after taking the rare higher ranks into account, the average is exactly $1/2$.

References

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