The Irrationality Exponent of π is 2

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In October 2026, OpenAI released a preprint (OpenAI 2026) claiming to resolve a long-standing problem in number theory: determining the *irrationality exponent* of π. The paper proves that this exponent equals 2, which means, roughly speaking, that π cannot be approximated by rational numbers substantially better than a typical irrational number. The result is accompanied by a formalization in Lean, although the preprint has not yet undergone the usual process of independent mathematical scrutiny.

In two recent posts, we discussed the irrationality of particular mathematical constants. However, once we know that a number is irrational, we can ask a more refined question: *how well can it be approximated by rational numbers?*

A classical theorem of Dirichlet states that, for every irrational real number α, there exist infinitely many integers m and positive integers n satisfying

$$0 < \left|\alpha - \frac{m}{n}\right| < \frac{1}{n^2}.$$

In 1891, Hurwitz (Hurwitz 1891) improved this result by proving that every irrational real number α admits infinitely many rational approximations satisfying

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{\sqrt{5}\,n^2}.$$

Moreover, the constant $\sqrt{5}$ is optimal: it cannot be replaced by any larger constant for all irrational numbers. For example, the *golden ratio*

$$\alpha = \frac{1+\sqrt{5}}{2}$$

does not admit infinitely many rational approximations satisfying the corresponding inequality with any larger constant.

Can we do better if we allow the constant to depend on α? In particular, can we replace n^2 in the denominator by a larger power of n?

For an irrational real number α, its *irrationality exponent* $\gamma(\alpha)$ is defined as the supremum of all real numbers $\gamma > 0$ such that

$$0 < \left|\alpha - \frac{m}{n}\right| < \frac{1}{n^{\gamma}} \tag{1}$$

has infinitely many solutions with $m \in \mathbb{Z}$ and n a positive integer. Here the supremum means the smallest upper bound for all such exponents γ.

Dirichlet's theorem implies that

$$\gamma(\alpha) \geq 2$$

for every irrational α.

However, some irrational numbers can be approximated much better than others. There even exist irrational numbers α such that $\gamma(\alpha) = \infty$, meaning that for every fixed $\gamma > 0$, inequality (1) has infinitely many solutions.

On the other hand, a classical theorem of Khintchine (Khintchine 1924) implies that

$$\gamma(\alpha) = 2$$

for *almost every* real number α. Here "almost every" means that the set of exceptions has Lebesgue measure zero, that is, zero length in the usual measure-theoretic sense.

Thus, although some exceptional irrational numbers admit extraordinarily accurate rational approximations, a typical irrational number cannot be approximated substantially better than the bound given by Dirichlet's theorem.

But what about specific famous irrational numbers, such as π?

Despite the fact that π has been known to be irrational since the eighteenth century, determining its irrationality exponent turned out to be extremely difficult. There was substantial progress in finding upper bounds. For example, in 2020, Zeilberger and Zudilin (Zeilberger and Zudilin 2020) proved that

$$\gamma(\pi) \leq 7.103205334137\ldots.$$

However, this was still far from the conjectured value 2.

The new OpenAI preprint (OpenAI 2026) claims to establish exactly this value.

Theorem 1 (OpenAI) *For every $\epsilon > 0$, there exists a positive integer $n_0(\epsilon)$ such that, for every integer $n \geq n_0(\epsilon)$ and every $m \in \mathbb{Z}$,*

$$\left|\pi - \frac{m}{n}\right| \geq \frac{1}{n^{2+\epsilon}}.$$

Consequently,

$$\gamma(\pi) = 2.$$

Thus, for any fixed $\epsilon > 0$, there can be only finitely many rational approximations to π satisfying

$$\left|\pi - \frac{m}{n}\right| < \frac{1}{n^{2+\epsilon}}.$$

In particular, no exponent greater than 2 is achievable infinitely often.

Notice that the theorem does not assert the existence of a constant $c > 0$ such that

$$\left|\pi - \frac{m}{n}\right| \geq \frac{c}{n^2}$$

for all integers m and positive integers n. This would be a stronger statement and remains open. The threshold $n_0(\epsilon)$ in the theorem is also not given explicitly.

Let us briefly discuss the proof. The authors argue by contradiction, assuming that π admits infinitely many exceptionally accurate rational approximations. They select several such approximations with rapidly increasing denominators and use them to construct a large square matrix with a nonzero determinant.

The determinant can then be estimated in two different ways. On the one hand, the arithmetic properties of its entries give a lower bound for its absolute value. On the other hand, the assumed accuracy of the rational approximations, combined with carefully chosen polynomial identities and properties of the exponential function, gives a much smaller upper bound. The two estimates contradict each other.

The main technical achievement is a new interpolation method that makes these estimates sufficiently strong to exclude every exponent greater than 2.

The theorem gives a remarkably precise description of how π can be approximated by rational numbers. We have known for centuries that π is irrational, but only now has a proof been proposed that its irrationality exponent is the smallest possible one.

References

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