

Catalan's Constant is Irrational

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In October 2026, OpenAI released a collection of mathematical preprints obtained using an internal AI model. One of these papers, dated September 24, 2026, is entitled *Catalan's constant is irrational* (OpenAI 2026). It claims to resolve a classical problem in number theory: whether a certain number defined by a very simple infinite series can be expressed as a ratio of two integers.

Recall that a real number is called *irrational* if it cannot be expressed as a fraction p/q , where p and q are integers and $q \neq 0$. Famous examples include $\sqrt{2}$, π , and e .

Many other naturally occurring mathematical constants have been studied for centuries, but even determining whether they are rational or irrational can be extremely difficult. In a recent post, we discussed a preprint concerning the irrationality of $\zeta(5)$, where

$$\zeta(5) = \sum_{n=1}^{\infty} \frac{1}{n^5}.$$

Another classical example is *Catalan's constant*, defined by

$$G = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2}.$$

In other words,

$$G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \dots \approx 0.9159655942.$$

The constant is named after the Belgian mathematician Eugène Catalan, who also formulated the famous Catalan conjecture about consecutive perfect powers. Despite the common name, the two problems are unrelated.

Catalan's constant appears naturally in many areas of mathematics. For example, it has the integral representation

$$G = \int_0^1 \frac{\arctan x}{x} dx,$$

where $\arctan x$ is the inverse tangent function. Indeed, this follows by

integrating the familiar power series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

term by term after dividing by x .

The arithmetic nature of G has remained mysterious for a very long time. To understand why, consider the more general *Dirichlet beta function*

$$\beta(s) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^s}, \quad s > 0.$$

Thus, $G = \beta(2)$.

For positive odd integers, the values of $\beta(s)$ are rational multiples of powers of π . For example,

$$\beta(1) = \frac{\pi}{4}, \quad \beta(3) = \frac{\pi^3}{32}.$$

However, no comparable formula is known for $\beta(2)$ or, more generally, for the values at positive even integers.

There has nevertheless been substantial progress. In 2003, Rivoal and Zudilin (Rivoal and Zudilin 2003) proved that infinitely many of the numbers

$$\beta(2), \beta(4), \beta(6), \dots$$

are irrational. They also proved that at least one of

$$\beta(2), \beta(4), \dots, \beta(14)$$

is irrational.

Such results are striking, but they do not tell us which particular numbers are irrational. In particular, they did not establish the irrationality of $G = \beta(2)$.

Another approach, developed by Zudilin (Zudilin 2003) and other mathematicians, was to construct especially good rational approximations to G . However, obtaining rational approximations with small error is not enough to prove irrationality. After all, every real number, including every rational number, can be approximated arbitrarily accurately by fractions.

The real challenge is to construct rational approximations whose errors are exceptionally small *relative to their denominators*. For example, suppose we can find integers a_n, b_n such that

$$0 < |a_n G - b_n| \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then G must be irrational. Indeed, if $G = p/q$ for some integers p, q with $q > 0$, then

$$q(a_n G - b_n) = a_n p - q b_n$$

is a nonzero integer, so

$$|a_n G - b_n| \geq \frac{1}{q},$$

a contradiction.

Constructing such approximations proved extremely difficult for Catalan's constant. The new OpenAI preprint (OpenAI 2026) uses a different method to establish the desired irrationality.

Theorem 1 (OpenAI) *Catalan's constant*

$$G = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^2}$$

is irrational.

The proof uses a sophisticated version of the *determinant method*. Instead of constructing individual rational approximations to G , the authors construct a sequence of determinants Δ_N of large square matrices, whose entries are rational expressions involving G .

The construction is designed so that, if G were rational, then every Δ_N would also be rational, with its denominator satisfying a sufficiently strong bound. Moreover, the authors prove that these determinants are nonzero for infinitely many values of N .

On the other hand, an integral representation of the same determinants gives a very small upper bound for their absolute values. The key achievement is to make this upper bound smaller than what is possible for a nonzero rational number with the previously established denominator bound.

Thus, assuming that G is rational leads to a contradiction.

The argument builds on earlier work on irrationality proofs using determinants, including that of Zudilin (Zudilin 2017). It combines carefully chosen polynomial identities, estimates for integrals, and detailed analysis of the prime factors appearing in denominators. The paper also provides explicit calculations needed to verify the final inequalities.

The OpenAI repository reports a formalization of the main irrationality statement in Lean, a computer proof assistant. The result is still presented as a preprint, and independent mathematical scrutiny remains important.

The theorem concerns irrationality, not *transcendence*. A number is called transcendental if it is not a root of any nonzero polynomial with integer coefficients. Every transcendental number is irrational, but the converse is false: for example, $\sqrt{2}$ is irrational but satisfies $x^2 - 2 = 0$.

Thus, even if the new proof is accepted, it will remain unknown whether Catalan's constant is transcendental. Nevertheless, establishing its irrationality would resolve one of the most famous longstanding problems about special values of infinite series.

References

OpenAI. 2026. *Catalan's Constant Is Irrational*. OpenAI Math Release preprint. <https://github.com/openai/math/blob/main/preprints/Catalans-constant-is-irrational-September-24-2026/paper.pdf>.

Rivoal, Tanguy, and Wadim Zudilin. 2003. "Diophantine Properties of Numbers Related to Catalan's Constant." *Mathematische Annalen* 326 (4): 705–21. <https://doi.org/10.1007/s00208-003-0420-2>.

Zudilin, Wadim. 2003. "An Apéry-Like Difference Equation for Catalan's Constant." *Electronic Journal of Combinatorics* 10 (1): R14. <https://doi.org/10.37236/1707>.

Zudilin, Wadim. 2017. "A Determinantal Approach to Irrationality." *Constructive Approximation* 45 (2): 301–10. <https://doi.org/10.1007/s00365-016-9333-7>.