

The Two-Point Chowla Conjecture

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On September 24, 2026, OpenAI released a preprint (OpenAI 2026) announcing a proof of the *two-point Chowla conjecture*, one of the most famous open problems in analytic number theory. The conjecture, formulated by Chowla in 1965, predicts that the parities of the numbers of prime factors of two consecutive positive integers behave independently. The paper establishes this prediction, with a quantitative estimate on the rate of convergence. The result is currently a preprint, accompanied by a formalization in Lean.

In a previous post, we discussed the history of this problem and the weaker, logarithmically averaged version of the conjecture, proved by Tao in 2016 and subsequently strengthened by Helfgott, Radziwiłł, and Pilatte. Let me briefly recall the question.

For a positive integer n , let $\Omega(n)$ denote the number of prime factors of n , counted with multiplicity. For example,

$$\Omega(12) = \Omega(2^2 \cdot 3) = 3.$$

The *Liouville function* is defined by

$$\lambda(n) = (-1)^{\Omega(n)}.$$

Thus, $\lambda(n) = 1$ if n has an even number of prime factors, and $\lambda(n) = -1$ otherwise.

It is natural to expect that the values of $\lambda(n)$ behave approximately like independent random signs. Indeed, a classical consequence of the prime number theorem is that

$$\sum_{n \leq X} \lambda(n) = o(X) \quad \text{as } X \rightarrow \infty.$$

Here $o(X)$ denotes a quantity whose ratio to X tends to zero. This estimate means that, among the positive integers not exceeding X , approximately half have an even number of prime factors and half have an odd number.

But what happens when we consider two consecutive integers n and $n + 1$? Is the parity of $\Omega(n)$ related to the parity of $\Omega(n + 1)$?

In 1965, Chowla (Chowla 1965) conjectured that

$$\sum_{n \leq X} \lambda(n)\lambda(n+1) = o(X) \quad \text{as } X \rightarrow \infty. \quad (1)$$

Indeed, the product $\lambda(n)\lambda(n+1)$ equals 1 when $\Omega(n)$ and $\Omega(n+1)$ have the same parity, and -1 when their parities differ. Therefore, (1) says that these two possibilities occur approximately equally often.

More generally, Chowla conjectured that for any fixed positive integer k and any distinct nonnegative integers $h_1, \dots, h_k$,

$$\sum_{n \leq X} \lambda(n+h_1) \cdots \lambda(n+h_k) = o(X).$$

The case $k=1$ follows from the prime number theorem. However, already the case $k=2$ resisted proof for more than sixty years.

In 2016, Tao (Tao 2016) proved a weaker version of (1), in which each summand is divided by n :

$$\sum_{n \leq X} \frac{\lambda(n)\lambda(n+1)}{n} = o(\log X).$$

This is called the *logarithmically averaged Chowla conjecture*. It gives greater weight to smaller integers, whereas the original conjecture counts every integer equally.

The distinction is important. The original conjecture implies the logarithmically averaged version, but the converse does not follow. Logarithmic averaging can conceal substantial correlations concentrated in particular ranges of large integers.

As discussed in my earlier post, Helfgott and Radziwiłł (Helfgott and Radziwiłł 2021) improved Tao's result, and Piatte (Piatte 2026) eventually established the bound

$$\sum_{n \leq X} \frac{\lambda(n)\lambda(n+1)}{n} = O((\log X)^{1-c})$$

for some constant $c > 0$.

However, the original conjecture (1), without logarithmic averaging, remained open.

The new OpenAI preprint (OpenAI 2026) proves a considerably stronger result, allowing the arguments of the Liouville functions to be arbitrary nonproportional linear expressions.

Theorem 1 (OpenAI) *There exists an absolute constant $c > 0$ such that, for any fixed positive integers a_1, a_2 and nonnegative integers b_1, b_2 satisfying*

$$a_1 b_2 - a_2 b_1 \neq 0,$$

we have

$$\sum_{n \leq X} \lambda(a_1 n + b_1) \lambda(a_2 n + b_2) = O\left(\frac{X}{(\log X)^c}\right) \quad \text{as } X \rightarrow \infty.$$

The constant implicit in the O -notation may depend on a_1, a_2, b_1, b_2 , but c is independent of these parameters.

Taking $a_1 = a_2 = b_2 = 1$ and $b_1 = 0$, we obtain

$$\sum_{n \leq X} \lambda(n) \lambda(n+1) = O\left(\frac{X}{(\log X)^c}\right),$$

which proves (1).

The theorem has a particularly nice interpretation. Let $E(X)$ denote the number of positive integers $n \leq X$ for which $\Omega(n)$ and $\Omega(n+1)$ have the same parity, and let $D(X)$ denote the number for which their parities differ. Then

$$E(X) - D(X) = \sum_{n \leq X} \lambda(n) \lambda(n+1).$$

Since $E(X) + D(X)$ equals the number of positive integers not exceeding X , the theorem implies

$$E(X) = \frac{X}{2} + O\left(\frac{X}{(\log X)^c}\right), \quad D(X) = \frac{X}{2} + O\left(\frac{X}{(\log X)^c}\right).$$

In fact, combining this result with the prime number theorem, we can conclude that each of the four possible combinations

$$(\lambda(n), \lambda(n+1)) = (1, 1), (1, -1), (-1, 1), (-1, -1)$$

occurs for

$$\left(\frac{1}{4} + o(1)\right) X$$

positive integers $n \leq X$. Thus, the parities of the numbers of prime factors of consecutive integers are asymptotically independent.

Let us briefly discuss the main idea of the proof. The authors build on methods developed by Helfgott, Radziwiłł, and Pilatte. These methods study graphs whose vertices are integers and whose edges connect integers separated by certain multiples of primes. The properties of these graphs can be used to estimate correlations between values of the Liouville function.

The main challenge is to obtain sufficiently strong cancellation without logarithmic averaging. The authors combine estimates for the Liouville function on short intervals with a more elaborate construction involving products of

primes. They then estimate the number of closed walks in the resulting graphs, where a closed walk is a sequence of adjacent vertices that starts and ends at the same vertex. This allows them to control the correlations on every sufficiently large scale and establish the stated power-of-logarithm saving.

The paper actually proves a more general result concerning correlations of *multiplicative functions*, that is, functions f satisfying

$$f(mn) = f(m)f(n)$$

whenever m and n are relatively prime. Under suitable assumptions excluding functions that imitate simple periodic or regularly oscillating patterns, the authors prove the ordinary two-point case of the corrected Elliott conjecture, a far-reaching generalization of Chowla's conjecture.

The result represents a major advance in our understanding of the apparent randomness of prime factorization. It establishes that the parity of the number of prime factors of n is asymptotically independent of that of $n + 1$, something that had been expected since 1965 but had remained out of reach.

The higher-point cases of Chowla's conjecture, involving products of three or more values of the Liouville function, remain open. Thus, even after this breakthrough, some of the deepest questions about the statistical behavior of prime factors are still unresolved.

References

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