

Kummer's Bias in Cubic Gauss Sums

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On September 25, 2026, OpenAI released a preprint (OpenAI 2026) claiming an unconditional proof of a conjecture formulated by Patterson in 1978 concerning *cubic Gauss sums*. The result explains a surprising numerical phenomenon first observed by Kummer in 1846: certain sums associated with prime numbers exhibit a small but persistent bias. The result remains a preprint, awaiting independent verification.

In number theory, one sometimes encounters an unexpected pattern that holds for all numbers examined experimentally but eventually disappears when sufficiently large numbers are considered. This phenomenon is sometimes called the *law of small numbers*.

A famous example concerns the prime-counting function $\pi(x)$, which counts the primes not exceeding x . Early numerical computations suggested that

$$\pi(x) < \text{Li}(x),$$

where $\text{Li}(x)$ is the *logarithmic integral*, defined for $x > 1$ by

$$\text{Li}(x) = \lim_{\varepsilon \rightarrow 0^+} \left(\int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^x \frac{dt}{\log t} \right).$$

Here and below, $\log$ denotes the natural logarithm. The limit is needed because $\log 1 = 0$, so the integrand is not defined at $t = 1$.

For a long time, mathematicians suspected that the inequality $\pi(x) < \text{Li}(x)$ might hold for every $x > 1$. However, in 1914, Littlewood (Littlewood 1914) proved that the difference $\pi(x) - \text{Li}(x)$ changes sign infinitely often. Thus, even extremely convincing numerical evidence can be misleading.

A similar phenomenon occurs with *cubic Gauss sums*. For every prime p such that $p \equiv 1 \pmod{3}$, meaning that $p - 1$ is divisible by 3, define

$$S_p = \sum_{n=1}^p \cos\left(\frac{2\pi n^3}{p}\right).$$

These sums arise naturally when studying the distribution of cubes modulo p , that is, the possible remainders of the numbers n^3 upon division by p .

A classical estimate shows that

$$|S_p| \leq 2\sqrt{p}.$$

Consequently, the *normalized cubic Gauss sum*

$$s_p = \frac{S_p}{2\sqrt{p}}$$

always belongs to the interval $[-1, 1]$.

In 1846, Kummer (Kummer 1846) studied these quantities numerically. He observed that, among primes $p < 500$ with $p \equiv 1 \pmod{3}$, the values of s_p belonged to the interval

$$I_+ = \left[\frac{1}{2}, 1\right]$$

approximately three times as often as to the symmetric interval

$$I_- = \left[-1, -\frac{1}{2}\right].$$

This was surprising, because there was no apparent reason for large positive values to be so much more frequent than large negative ones.

As computers became available, mathematicians examined much larger primes and discovered that this striking imbalance gradually disappeared. In fact, Heath-Brown and Patterson (Heath-Brown and Patterson 1979) proved in 1979 that, as we consider more and more primes, the proportions of normalized Gauss sums belonging to I_+ and I_- approach the same limit.

One might therefore conclude that Kummer's observed bias was simply an illusion caused by examining too few primes. However, Patterson (Patterson 1978) discovered that the situation is more subtle. Although the large imbalance in the frequencies disappears, there should still be a small systematic bias in the *average* of the normalized sums.

More precisely, Patterson predicted that

$$\sum_{\substack{p \leq X \\ p \equiv 1 \pmod{3}}} s_p$$

should have a positive main term proportional to

$$\frac{X^{5/6}}{\log X}.$$

Thus, even though large positive and large negative values become equally frequent, their contributions to the sum do not cancel completely.

Proving this prediction turned out to be extremely difficult. In 2024, Dunn and Radziwiłł (Dunn and Radziwiłł 2024) proved a related asymptotic formula in which the primes are counted with smoothly varying weights rather than an

abrupt cutoff at X . Their proof assumed a version of the *Generalized Riemann Hypothesis*, a famous unproved conjecture about zeros of certain complex functions used to study prime numbers. Their paper appeared in the *Annals of Mathematics*.

The new OpenAI preprint (OpenAI 2026) claims to remove this assumption and establish an exact asymptotic formula for the original sum.

Theorem 1 (OpenAI) As $X \rightarrow \infty$,

$$\sum_{\substack{p \leq X \\ p \equiv 1 \pmod{3}}} \frac{S_p}{2\sqrt{p}} \sim \frac{(2\pi)^{2/3}}{5\Gamma(2/3)} \frac{X^{5/6}}{\log X},$$

where

$$S_p = \sum_{n=1}^p \cos\left(\frac{2\pi n^3}{p}\right)$$

and Γ denotes the gamma function, defined for $z > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Here the notation $f(X) \sim g(X)$ means that the ratio $f(X)/g(X)$ tends to 1 as $X \rightarrow \infty$.

The constant

$$C = \frac{(2\pi)^{2/3}}{5\Gamma(2/3)}$$

is positive. Thus, the theorem gives a precise quantitative description of the bias predicted by Patterson.

Why does this not contradict the fact that large positive and large negative values become equally frequent?

The explanation is that the bias is very small compared with the total number of primes. A classical result called the *prime number theorem for arithmetic progressions* states, in particular, that

$$\#\{p \leq X : p \equiv 1 \pmod{3}\} \sim \frac{X}{2\log X}.$$

In other words, approximately half of all primes belong to the residue class 1 modulo 3.

Consequently, dividing the sum in the theorem by the number of primes being counted, we obtain

$$\frac{1}{\#\{p \leq X : p \equiv 1 \pmod{3}\}} \sum_{\substack{p \leq X \\ p \equiv 1 \pmod{3}}} s_p \sim 2CX^{-1/6}.$$

Since $X^{-1/6}$ tends to zero, the average of the normalized Gauss sums also tends to zero. However, it approaches zero in a systematic way, with a positive main term of a precisely determined size.

Notice that a positive average does not necessarily mean that positive values occur more frequently than negative ones. It is possible, for example, that positive values are slightly larger in magnitude even when positive and negative values occur equally often. The theorem concerns the sum of the normalized Gauss sums, not the difference between the number of positive and negative terms.

Let us briefly discuss the main idea of the proof. The challenge is to distinguish a very small systematic bias from much larger fluctuations in the individual sums. The authors transform the problem into estimating several complicated sums involving primes. They identify the contribution responsible for the positive main term and develop new estimates showing that the remaining contributions are sufficiently small. This allows them to obtain the predicted formula without the unproved hypothesis needed in the earlier work.

The result provides a beautiful illustration of the law of small numbers. Kummer's original observation was misleading in one sense: the dramatic imbalance between large positive and large negative values eventually disappears. Yet it also contained a grain of truth: a much smaller positive bias persists, and its precise size is given by Patterson's prediction.

References

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