

Van Vu's talk on random matrix theory

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<https://functor.network/user/3333/entry/2016>

This post continues our series of posts about online seminars devoted to accessible presentations of some of the most significant mathematical theorems of the twenty-first century.

The next lecture will take place on *Wednesday, November 4, 2026, at 12:00 noon (UK time)*. Prof. Van Vu (The University of Hong Kong, China) will speak on the topic *Current directions in random matrix theory*.

Prof. Van Vu is one of the world's leading experts in random matrix theory, probability, and combinatorics. He has made fundamental contributions to our understanding of the eigenvalues, invertibility, and other properties of random matrices, including landmark results on the circular law and universality.

The talk will give an accessible overview of some current directions in random matrix theory, focusing on the remarkable phenomenon that many properties of large random matrices do not depend on the precise distribution of their entries. It will also discuss recent developments, open problems, and connections with combinatorics, probability, statistics, and data science.

To join the talk, please click [here](#) at the scheduled time. No registration is required.

For announcements of other talks, as well as descriptions of recent major mathematical breakthroughs, see the full list of posts on this blog.

Let me now give some background on the mathematics of the talk, concentrating on several major results of Van Vu.

A *random matrix* is a matrix whose entries are random variables. One of the simplest models is an $n \times n$ matrix M_n whose entries are independent and identically distributed, each equal to $+1$ or -1 with probability $1/2$. There are 2^{n^2} such matrices, and we select one of them uniformly at random.

An elementary but surprisingly difficult question is: what is the probability that M_n is *singular*, that is, has determinant zero?

For example, M_n is singular whenever two of its rows or columns are equal up to sign. This gives the lower bound

$$\mathbb{P}(\det(M_n) = 0) \geq (1 + o(1))n^2 2^{1-n}.$$

It was conjectured that this is asymptotically the correct answer.

In 2006, Tao and Vu (Tao and Vu 2006) obtained the bound

$$\mathbb{P}(\det(M_n) = 0) = O(0.958^n),$$

which they improved in 2007 to

$$\mathbb{P}(\det(M_n) = 0) \leq \left(\frac{3}{4} + o(1)\right)^n$$

in a celebrated paper (Tao and Vu 2007). Their proof introduced powerful methods from additive combinatorics into random matrix theory. I discussed this theorem in detail in an earlier blog post.

In 2010, Bourgain, Vu, and Wood (Bourgain et al. 2010) improved the bound further to

$$\mathbb{P}(\det(M_n) = 0) \leq \left(\frac{1}{\sqrt{2}} + o(1)\right)^n.$$

Subsequent developments culminated in the determination of the optimal exponential rate by Tikhomirov in 2020. Very recently, Hu, Song, and Wu (Hu et al. 2026) posted a preprint proving the sharper asymptotic formula

$$\mathbb{P}(\det(M_n) = 0) = (1 + o(1))n^2 2^{1-n},$$

resolving the original conjecture. See this recent post for more details.

Another important model is that of *random symmetric matrices*, in which the entries on and above the main diagonal are independent random variables, while the remaining entries are determined by the symmetry condition $a_{ij} = a_{ji}$. This dependence between entries makes the analysis more difficult.

In 2006, Costello, Tao, and Vu (Costello et al. 2006) proved that a random symmetric matrix with Bernoulli entries is non-singular with probability tending to 1. More precisely, they obtained the upper bound

$$\mathbb{P}(\det(M_n) = 0) = O(n^{-1/8+\epsilon})$$

for every fixed $\epsilon > 0$. Their proof developed a quadratic version of the Littlewood–Offord inequalities, which estimate the probability that a random sum takes a prescribed value.

For many applications, it is important not only that a matrix is invertible, but also that its inverse is not too large. Recall that the *norm* of a matrix A is

$$\|A\| = \sup_{x \neq 0} \frac{|Ax|}{|x|},$$

where $|\cdot|$ denotes the Euclidean norm. If A is invertible, the quantity $\|A^{-1}\|$ measures how sensitive the solution of the linear system $Ax = b$ is to small changes in b .

In 2009, Tao and Vu (Tao and Vu 2009) proved that, for every constant $C > 0$, there exists a constant $B > 0$ such that

$$\mathbb{P}(\|M_n^{-1}\| \leq n^B) \geq 1 - n^{-C}$$

for all sufficiently large n , where $\|M_n^{-1}\| = \infty$ if M_n is singular.

This result followed from their development of *inverse Littlewood–Offord theory*, which explains why a random signed sum can have unusually large probability of taking a particular value. These ideas have since become central tools in random matrix theory.

The norm of the inverse is the reciprocal of the smallest singular value $s_{\min}(M_n)$. In 2010, Tao and Vu (Tao and Vu 2010a) proved a remarkable *universality theorem*: under suitable finite-moment assumptions on the entries, the limiting probability distribution of

$$n s_{\min}(M_n)^2$$

is the same as for matrices with independent Gaussian entries. Thus, even the behavior of the smallest singular value is largely independent of the precise distribution of the matrix entries. Their result also gave nearly optimal estimates related to a conjecture of Spielman and Teng.

Universality is an even more striking phenomenon when we consider the *eigenvalues* of random matrices. Recall that a complex number λ is an eigenvalue of a square matrix A if

$$Av = \lambda v$$

for some nonzero vector v .

Let A_n be an $n \times n$ random matrix whose entries are independent and identically distributed complex random variables with mean zero and variance one. The eigenvalues of A_n need not be real. However, after dividing the matrix by $\sqrt{n}$, they exhibit an astonishingly simple limiting distribution.

In 2010, Tao and Vu (Tao and Vu 2010b), with an appendix by Manjunath Krishnapur, proved the general form of the celebrated *circular law conjecture*. It states that, as $n \rightarrow \infty$, the eigenvalues of

$$\frac{1}{\sqrt{n}} A_n$$

become uniformly distributed in the unit disk

$$\{z \in \mathbb{C} : |z| \leq 1\}.$$

More precisely, the proportion of eigenvalues in any sufficiently regular region converges almost surely to the area of that region intersected with the unit disk, divided by π .

The remarkable feature of this result is that it requires no assumptions on the distribution of the entries beyond independence, zero mean, and unit variance. Whether the entries are Gaussian, discrete, or have some other distribution, the same circular law emerges in the limit.

The circular law describes the global distribution of eigenvalues. However, it does not tell us how the eigenvalues are arranged locally, for example, what the typical distance is between two consecutive eigenvalues.

This leads to another major achievement of Tao and Vu, the *Four Moment Theorem* (Tao and Vu 2011). Consider a random *Hermitian matrix*, that is, a complex matrix A satisfying

$$A = A^*,$$

where A^* is the conjugate transpose. Such a matrix has only real eigenvalues, which can be arranged in increasing order

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

We may then study the gaps

$$\lambda_{i+1} - \lambda_i$$

between consecutive eigenvalues.

Tao and Vu proved that, under suitable regularity assumptions, many local properties of eigenvalues are determined by only the first four moments of the distribution of the matrix entries. In particular, they established universality results for the distribution of gaps between consecutive eigenvalues. This was a major step towards understanding why very different random matrix models exhibit the same local statistical behavior.

For non-Hermitian matrices, the eigenvalues may be complex, but a related universality phenomenon holds. In 2015, Tao and Vu (Tao and Vu 2015) proved that, under appropriate assumptions, local eigenvalue statistics again depend only on the first four moments of the entries.

One particularly interesting consequence concerns the number of *real eigenvalues* of a random real matrix. If the entries are independent, have sufficiently fast decay, and their first four moments agree with those of a standard Gaussian random variable, then Tao and Vu proved that the number of

real eigenvalues is

$$\sqrt{\frac{2n}{\pi}} + o(\sqrt{n})$$

with probability tending to 1 as $n \rightarrow \infty$.

Thus, although a real $n \times n$ matrix can have anywhere between 0 and n real eigenvalues (subject to the usual parity restriction), a large random matrix typically has only about

$$0.798\sqrt{n}$$

real eigenvalues.

These results illustrate a central theme of modern random matrix theory: complicated random systems often exhibit universal behavior governed by a surprisingly small amount of information about their components. Prof. Van Vu has been one of the leading figures in developing this theory. The forthcoming lecture will provide an opportunity to learn about the current state of this fascinating subject and the challenges that remain.

References

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