

The Gaussian moat conjecture

Bogdan Grechuk · 17 Oct 2026

<https://functor.network/user/3333/entry/2014>

On September 26, 2026, OpenAI (OpenAI 2026) released a preprint presenting a proof of the Gaussian moat conjecture. The result answers a question that is easy to visualize: can one walk to infinity using Gaussian primes as stepping stones, while keeping every step below some fixed length?

We have already discussed small gaps and long gaps between ordinary primes. On the real line, such a walk is impossible because there are arbitrarily large gaps between consecutive primes. Indeed, for every integer $m \geq 2$, all the numbers

$$m! + 2, m! + 3, \dots, m! + m$$

are composite. These gaps eventually block any journey with a fixed bound on its step lengths.

What happens if we replace the real line by the complex plane?

The *Gaussian integers* are the complex numbers $a + bi$, where a and b are integers and $i^2 = -1$. Their units, meaning the elements whose multiplicative inverses are also Gaussian integers, are $1, -1, i, -i$. A nonzero Gaussian integer which is not a unit is a *Gaussian prime* if it cannot be written as a product of two Gaussian integers neither of which is a unit.

For example, 5 is an ordinary prime but not a Gaussian prime, because

$$5 = (2 + i)(2 - i).$$

Both $2 + i$ and $2 - i$ are Gaussian primes. More generally, when a and b are both nonzero, $a + bi$ is a Gaussian prime if and only if $a^2 + b^2$ is an ordinary prime. On the coordinate axes, the Gaussian primes are $p, -p, pi, -pi$, where p is a positive prime satisfying $p \equiv 1 \pmod{4}$.

We view $a + bi$ as the point (a, b) in the plane and measure distances in the usual Euclidean way. Imagine that these prime points are stepping stones surrounded by water. Can we choose a fixed number $D > 0$ and an infinite sequence of distinct Gaussian primes

$$z_1, z_2, z_3, \dots$$

such that

$$|z_{n+1} - z_n| \leq D$$

for every $n \geq 1$?

This is the *Gaussian moat problem*, posed by Basil Gordon in 1962. An accessible account of its history and early progress is given by Gethner, Wagon, and Wick (Gethner et al. 1998).

The extra dimension makes a substantial difference. A long gap on a line blocks the way, but a large region without Gaussian primes need not stop a walk in the plane: the walker may simply go around it. To rule out every possible route, one needs barriers that cannot be bypassed.

The new preprint proves the following stronger statement. For $D > 0$, form a graph G_D whose vertices are all Gaussian primes, joining two distinct vertices whenever their distance is at most D . A connected component consists of all primes reachable from a given prime by finite walks along these edges.

Theorem. *For every $D > 0$, there is a finite integer B_D such that every connected component of G_D has at most B_D vertices.*

Consequently, any walk through distinct Gaussian primes with steps of length at most D visits at most B_D primes. The same bound works for every starting prime, including primes on the coordinate axes. This uniformity is stronger than merely ruling out infinite walks: finite connected components could, in principle, have arbitrarily large sizes.

The proof uses divisibility to construct a repeating pattern of obstacles. After choosing finitely many Gaussian primes, remove all their multiples. Apart from finitely many exceptions, every Gaussian prime survives. The difficult step is to choose the divisors so that the surviving lattice points admit no infinite walk with steps at most D . The argument combines geometric estimates with information theory. Periodicity then gives a uniform bound on component sizes, and restoring the exceptional primes preserves a finite bound.

The bound B_D is nonexplicit: the proof establishes its existence without giving a numerical formula in terms of D .

References

Gethner, Ellen, Stan Wagon, and Brian Wick. 1998. "A Stroll Through the Gaussian Primes." *The American Mathematical Monthly* 105 (4): 327–37. <https://doi.org/10.1080/00029890.1998.12004889>.

OpenAI. 2026. *Bounded-Step Walks on Gaussian Primes*. OpenAI Math Release preprint. <https://github.com/openai/math/blob/main/preprints/Bounded-Step-Walks-on-Gaussian-Primes-September-26-2026/paper.pdf>.