

Square-free values of quartic polynomials

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On September 24, 2026, OpenAI released a preprint (OpenAI 2026) containing a proof of a long-standing conjecture about square-free values of quartic polynomials. In particular, it answers a question posed by Erdős in 1953: are there infinitely many positive integers n such that $n^4 + 2$ is square-free? The new result is considerably stronger: a positive proportion of all positive integers n have this property.

Recall that an integer is called *square-free* if it is not divisible by the square of any prime. For example, $30 = 2 \cdot 3 \cdot 5$ is square-free, while $12 = 2^2 \cdot 3$ is not. Negative integers are allowed, but zero is not square-free.

A classical result in number theory states that the proportion of square-free integers among all positive integers tends to

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2} \approx 0.60793.$$

Indeed, for each prime p , exactly one of the p^2 residue classes modulo p^2 consists of multiples of p^2 . The formula expresses the idea that avoiding divisibility by p^2 for all primes should have the same effect as imposing these conditions independently.

What happens if, instead of considering all positive integers, we restrict attention to values of a polynomial? More precisely, given a polynomial $f \in \mathbb{Z}[x]$, how many of the numbers

$$f(1), f(2), f(3), \dots$$

are square-free?

This question has a long history. The quadratic case was investigated by Estermann (Estermann 1931) in 1931, and in 1953 Erdős (Erdős 1953) proved that cubic polynomials without repeated roots take infinitely many square-free values, provided there is no obvious divisibility obstruction. Hooley (Hooley 1967) subsequently established the corresponding asymptotic density.

For polynomials of degree four, however, the general problem remained open. Even for the simple polynomial

$$f(n) = n^4 + 2,$$

it was not known whether infinitely many values are square-free. Erdős explicitly raised this question in (Erdős 1953).

There is an obvious necessary condition. If some prime square p^2 divides $f(n)$ for every integer n , then none of the values can be square-free. We therefore assume that no such prime exists.

To formulate the expected answer, for every positive integer m define

$$\rho_f(m) = \#\{0 \leq a < m : f(a) \equiv 0 \pmod{m}\}.$$

Thus, $\rho_f(p^2)$ counts the residue classes modulo p^2 for which $f(n)$ is divisible by p^2 . The proportion of residue classes avoiding this divisibility is

$$1 - \frac{\rho_f(p^2)}{p^2}.$$

This suggests that the density of integers n for which $f(n)$ is square-free should be

$$c_f = \prod_{p \text{ prime}} \left(1 - \frac{\rho_f(p^2)}{p^2}\right).$$

The new result establishes exactly this formula for all irreducible quartic polynomials. Recall that a polynomial is *irreducible over $\mathbb{Q}$* if it cannot be written as a product of two polynomials of positive degree with rational coefficients.

Theorem 1 (OpenAI) *Let $f \in \mathbb{Z}[x]$ be a polynomial of degree 4, irreducible over $\mathbb{Q}$. Suppose that no prime square divides every integer value of f , or equivalently,*

$$\rho_f(p^2) < p^2 \quad \text{for every prime } p.$$

Then

$$\lim_{X \rightarrow \infty} \frac{\#\{1 \leq n \leq X : f(n) \text{ is square-free}\}}{X} = c_f,$$

where

$$c_f = \prod_{p \text{ prime}} \left(1 - \frac{\rho_f(p^2)}{p^2}\right) > 0.$$

In particular, the theorem applies to $f(n) = n^4 + 2$. This polynomial is irreducible by Eisenstein's criterion at the prime 2. Moreover,

$$f(0) = 2, \quad f(1) = 3,$$

so no prime square can divide all its values. Consequently, the theorem proves that

$$\#\{1 \leq n \leq X : n^4 + 2 \text{ is square-free}\} = (c_f + o(1))X$$

for some constant $c_f > 0$. Thus, not only are there infinitely many such integers, but they form a positive proportion of all positive integers.

Another particularly elegant example is

$$f(n) = n^4 + 1.$$

This polynomial is irreducible because it is the eighth cyclotomic polynomial. In this case, the constant c_f can be written especially simply.

Indeed, for an odd prime p , the congruence

$$n^4 + 1 \equiv 0 \pmod{p}$$

has solutions if and only if $p \equiv 1 \pmod{8}$. In that case it has exactly four solutions, each of which lifts uniquely to a solution modulo p^2 . Also, $n^4 + 1$ is never divisible by 4. Hence

$$c_f = \prod_{\substack{p \text{ prime} \\ p \equiv 1 \pmod{8}}} \left(1 - \frac{4}{p^2}\right).$$

The authors show that this product is greater than $3/4$. Therefore,

$$\#\{1 \leq n \leq X : n^4 + 1 \text{ is square-free}\} = (c_f + o(1))X, \quad c_f > \frac{3}{4}.$$

In other words, more than three quarters of all positive integers n have the property that $n^4 + 1$ is square-free, in the asymptotic sense.

Let us briefly discuss the proof. The main difficulty is to control large primes p such that

$$p^2 \mid f(n).$$

For small primes, the problem can be handled using congruences and the Chinese remainder theorem. However, as the degree of f increases, it becomes increasingly difficult to prove that the contribution from large prime squares is negligible.

For a quartic polynomial, $f(n)$ typically has size comparable to n^4 , so a prime whose square divides $f(n)$ can be as large as a constant multiple of n^2 . One must therefore estimate the number of solutions of equations of the form

$$f(n) = mp^2$$

with p large.

The proof combines factorization in algebraic number fields with a sophisticated version of the *determinant method*, which is used to bound the number of integer solutions of polynomial equations. The key achievement is a sufficiently strong upper bound for the number of n for which $f(n)$ is divisible by the square of a prime larger than n .

Once this estimate is established, the remaining primes can be treated by a sieve argument, leading to the predicted Euler-product density.

In fact, the paper proves more than the quartic result. An integer is called *k*-power-free if it is not divisible by the k th power of any prime. The authors establish the analogous density theorem for irreducible polynomials of degrees $4 \leq d \leq 8$, with $k = d - 2$. Combining this with earlier work of Browning (Browning 2011) gives the result for every degree $d \geq 4$: provided there is no fixed prime $(d - 2)$ th-power divisor, a positive proportion of the values of an irreducible degree- d polynomial are $(d - 2)$ -power-free.

This is another striking example of mathematical research carried out by AI. According to the OpenAI release, the result was obtained by an internal model, and the main theorem has also been formalized in Lean. The manuscript is currently a preprint, but its claimed resolution of a problem going back more than seventy years makes it particularly noteworthy.

References

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