

The singularity conjecture is proved

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In September 2026, Xuanang Hu, Zeyan Song, and Xinglong Wu posted a preprint (Hu et al. 2026) resolving a long-standing conjecture about the probability that a random matrix is singular. Remarkably, they proved that almost every singular random matrix with independent entries equal to $+1$ or -1 is singular for the simplest possible reason: two of its rows or columns are equal up to sign.

In a previous post, I discussed the history of this problem and the progress towards its resolution. Let me briefly recall the question.

An $n \times n$ matrix A is called *singular* if its determinant is zero, or, equivalently, if its rows are linearly dependent. Let M_n be a random $n \times n$ matrix whose entries are independent random variables, each equal to $+1$ or -1 with probability $1/2$. Put

$$P_n = \mathbb{P}(\det(M_n) = 0).$$

There are several obvious ways for M_n to be singular. For example, two rows may be equal or opposite, and the same can happen with two columns. For any fixed pair of rows, the probability that they are equal up to sign is 2^{1-n} . Since there are $\binom{n}{2}$ pairs of rows and the same number of pairs of columns, a simple inclusion–exclusion argument gives

$$P_n \geq (1 + o(1)) n^2 2^{1-n}.$$

It has long been conjectured that these elementary coincidences account for almost all singular matrices, so that

$$P_n = (1 + o(1)) n^2 2^{1-n}. \tag{1}$$

As discussed in my earlier post, this problem attracted considerable attention over several decades. In 2020, Tikhomirov (Tikhomirov 2020) established the optimal exponential rate

$$P_n = \left(\frac{1}{2} + o(1)\right)^n.$$

However, this estimate is much weaker than (1): it determines the exponential rate of decay but does not detect the polynomial factor n^2 .

In fact, the conjecture is a special case of a more general question. Let ξ be any non-constant real-valued random variable with finite support, and let $A_{n,\xi}$ be an $n \times n$ random matrix whose entries are independent copies of ξ .

In this setting, there are three obvious reasons for singularity: a row or column may consist entirely of zeros, two rows may be equal or opposite, or two columns may be equal or opposite. It was conjectured that these elementary events account for essentially all singular matrices.

In 2021, Jain, Sah, and Sawhney (Jain et al. 2021) confirmed this conjecture whenever ξ is not uniformly distributed on its support. Surprisingly, the remaining uniform case, which includes the classical random sign matrices, turned out to be particularly difficult.

Hu, Song, and Wu have now resolved this remaining case, completing the conjecture for every finitely supported real distribution.

Theorem 1 (Hu–Song–Wu, completing Jain–Sah–Sawhney) *Let ξ be a non-constant real-valued random variable with finite support, and let $A_{n,\xi}$ be an $n \times n$ random matrix with independent entries distributed as ξ . Then*

$$\begin{aligned} \mathbb{P}(\det(A_{n,\xi}) = 0) &= \mathbb{P}(\text{zero row or column}) \\ &\quad + (1 + o(1)) \mathbb{P}(\text{two equal or opposite rows or columns}) \end{aligned}$$

as $n \rightarrow \infty$.

In particular, if ξ takes the values -1 and $+1$ with probability $1/2$ each, then

$$P_n = (1 + o(1)) n^2 2^{1-n}.$$

Thus, conditioned on a large random matrix being singular, the probability that it has a zero row or column, or two rows or columns equal up to sign, tends to 1.

In other words, more complicated linear dependencies become negligible compared with these simple coincidences.

The result also applies to other distributions. For example, if all entries are independently and uniformly chosen from $\{0, 1\}$, then

$$\mathbb{P}(\det(A_{n,\xi}) = 0) = (1 + o(1)) n^2 2^{-n}.$$

Notice the difference by a factor of 2 from the random sign case. This arises because two opposite nonzero rows are possible for matrices with entries in $\{-1, 1\}$, but not for matrices with entries in $\{0, 1\}$.

The proof builds on techniques developed in the earlier work, especially Tikhomirov's method of *inversion of randomness*. The main difficulty is to rule out the enormous number of possible linear dependencies other than the elementary ones. Such dependencies can be expressed through random linear

combinations

$$a_1\xi_1 + \cdots + a_n\xi_n,$$

where the ξ_i are independent copies of ξ . Estimating the probability that these sums vanish is a central problem in Littlewood–Offord theory.

Hu, Song, and Wu develop new Fourier-analytic averaging estimates, combined with high-moment arguments, that allow them to count the exceptional dependencies with much greater precision than before. Earlier methods were sufficient to determine the exponential rate of singularity, but the new estimates are precise enough to recover the polynomial factor in (1).

In fact, their result is stronger than the asymptotic formula alone. If the entries are uniform on any fixed set of $q \geq 2$ real numbers, they show that, for every $\varepsilon > 0$, the probability of singularity without any of the elementary dependencies is

$$O\left(n^{1+\varepsilon}q^{-n}\right).$$

This is much smaller than the probability of elementary singularity, which has order n^2q^{-n} . Their methods also yield quantitative bounds for the smallest singular value of a random matrix, measuring how close it is to being singular.

The theorem therefore completes a classical problem in random matrix theory: for large matrices with independent entries from any fixed finite distribution, singularity is overwhelmingly explained by the simplest possible linear dependencies.

References

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