

The KLS conjecture is proved

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On 4 October 2026, Pierre Bizeul, Boaz Klartag, and Joseph Lehec posted a proof of the Kannan–Lovász–Simonovits (KLS) conjecture (Bizeul et al. 2026). Proposed in 1995, the conjecture predicts a universal principle of concentration for high-dimensional convex bodies. Its resolution explains why many quantities associated with a random point in such a body are much more predictable than the point itself.

A *convex body* is a compact set $K \subseteq \mathbb{R}^n$ with non-empty interior which contains the line segment joining any two of its points. Let X be a point chosen uniformly at random from K .

Before comparing different bodies, we must fix their position and scale. A very long, thin ellipsoid, for example, has much larger fluctuations along its longest axis than along its shortest one. An invertible affine transformation removes this imbalance. We say that X is *isotropic* if

$$\mathbb{E}X = 0, \quad \mathbb{E}\langle X, \theta \rangle^2 = 1 \quad \text{for every } \theta \in \mathbb{R}^n \text{ with } \|\theta\|_2 = 1.$$

Here $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ is the usual inner product and $\|x\|_2 = \sqrt{\langle x, x \rangle}$ is the Euclidean norm. Thus the centre of mass is at the origin and every direction has variance 1. Every convex body can be put in this position.

The theorem applies to a larger family of distributions. A probability density g is *log-concave* if $\{x : g(x) > 0\}$ is convex and $-\log g$ is convex there. Uniform distributions in convex bodies and Gaussian distributions are examples. We call a random vector log-concave when it has such a density, and use the same definition of isotropy as above.

For a real random variable Z , its variance

$$\text{Var}(Z) = \mathbb{E}(Z - \mathbb{E}Z)^2$$

measures its fluctuations about its average. The analytic formulation of KLS compares the fluctuations of $f(X)$ with how rapidly f changes.

Theorem 1 (KLS theorem) *There is a universal constant $C > 0$ such that, for every isotropic log-concave random vector $X \in \mathbb{R}^n$ and every continuously differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ for which the expectations below are finite,*

$$\text{Var}(f(X)) \leq C \mathbb{E} \|\nabla f(X)\|_2^2. \quad (1)$$

The constant is independent of the dimension, the distribution, and the function.

Here $\nabla f = (\partial f / \partial x_1, \dots, \partial f / \partial x_n)$ is the gradient. The inequality also extends to Lipschitz functions, whose gradients exist almost everywhere.

For a linear function $f(x) = \langle x, \theta \rangle$ with $\|\theta\|_2 = 1$, isotropy gives

$$\text{Var}(f(X)) = 1 \quad \text{and} \quad \mathbb{E} \|\nabla f(X)\|_2^2 = 1.$$

Thus linear functions already require a constant of at least 1. KLS says that arbitrary functions require only a universal factor more. No increasingly complicated function in an increasingly high dimension can make the ratio in (1) arbitrarily large.

To see the concentration meaning, call f *1-Lipschitz* if

$$|f(x) - f(y)| \leq \|x - y\|_2 \quad \text{for all } x, y \in \mathbb{R}^n.$$

Such a function changes by at most the distance moved. Examples include $f(x) = \|x\|_2$ and the distance from x to any fixed non-empty set. Its gradient has norm at most 1 almost everywhere, so KLS gives $\text{Var}(f(X)) \leq C$.

Log-concavity turns this bound, valid for every 1-Lipschitz function, into a much stronger probability estimate. By the equivalence between concentration and functional inequalities established by Emanuel Milman (Milman 2009), there are universal constants $C_0, c > 0$ such that

$$\mathbb{P}(|f(X) - \mathbb{E}f(X)| \geq t) \leq C_0 e^{-ct}, \quad t > 0. \quad (2)$$

The probability of a large deviation decreases exponentially, on the same scale in every dimension.

There is also a geometric interpretation. Imagine cutting an isotropic convex body into two parts, each containing a substantial proportion of its volume. KLS says that the area of the separating surface cannot be arbitrarily small compared with the smaller part's volume. In their original formulation, Kannan, Lovász, and Simonovits conjectured that hyperplane cuts already give the smallest such ratio up to a universal factor (Kannan et al. 1995). Allowing a curved separating surface improves the ratio by at most that factor. For log-concave distributions, this geometric statement and (1) are equivalent, with possibly different universal constants (Milman 2009).

The theorem also unifies two other famous problems in convex geometry.

First, isotropy gives $\mathbb{E}\|X\|_2^2 = n$, so the natural distance from the origin is $\sqrt{n}$. Applying (1) to $f(x) = \|x\|_2^2$, whose gradient is $2x$, gives

$$\text{Var}(\|X\|_2^2) \leq C \mathbb{E}\|2X\|_2^2 = 4Cn.$$

This is the *variance conjecture*. It immediately implies the *thin-shell conjecture*, since

$$\mathbb{E}(\|X\|_2 - \sqrt{n})^2 = \mathbb{E} \frac{(\|X\|_2^2 - n)^2}{(\|X\|_2 + \sqrt{n})^2} \leq \frac{\text{Var}(\|X\|_2^2)}{n} \leq 4C.$$

For any prescribed probability close to 1, a shell of width independent of n contains that proportion of the mass, while its radius grows as $\sqrt{n}$. Klartag and Lehec had already proved the variance bound in 2025 (Klartag and Lehec 2025b). KLS now supplies uniform concentration for every Lipschitz function, including quantities other than the radius.

Second, a universal thin-shell bound implies *Bourgain's slicing conjecture*. This asks whether every convex body of volume 1 has a hyperplane section whose $(n-1)$ -dimensional volume is at least a universal positive constant. Eldan and Klartag proved the implication from thin shells (Eldan and Klartag 2011). The slicing problem itself was settled by Klartag and Lehec in 2025, before the thin-shell theorem (Klartag and Lehec 2025a). Thus these three conjectures were resolved in the reverse order of this chain of implications:

$$\text{KLS} \implies \text{variance and thin shells} \implies \text{slicing}.$$

Much of the progress towards KLS came from *stochastic localization*, introduced by Ronen Eldan (Eldan 2013). This method studies a probability distribution through random perturbations that gradually concentrate its mass. Successive refinements reduced the dependence on dimension in the best known Poincaré bound, where the Poincaré constant is the smallest possible coefficient in (1) for a fixed distribution.

In July 2026, Brayden Letwin obtained a bound of order $\sqrt{\log n}$ for this constant (Letwin 2026). On 1 October, Zhao Song and Xinzhi Zhang reduced it to

$$C 16^{\log^*(n+2)},$$

where $\log^* x$ is the number of successive natural logarithms needed to bring x to at most 1 (Song and Zhang 2026b). This grows extraordinarily slowly, but still grows without bound. The new theorem removes the dependence on dimension altogether. Song and Zhang also posted a revised version of their paper on 4 October which contains a proof of the full conjecture (Song and Zhang 2026a).

The Bizeul–Klartag–Lehec proof uses a criterion from the first version of Song and Zhang’s paper. For a small vector z , consider the *tilt average*

$$F_f(z) = \frac{\mathbb{E}[f(X)e^{\langle z, X \rangle}]}{\mathbb{E}e^{\langle z, X \rangle}}.$$

The exponential factor gives more weight to points in the direction of z . The derivatives of F_f describe how the average of f responds to this change. Song and Zhang’s criterion converts suitable bounds on derivatives of all orders into a Poincaré bound. Bizeul, Klartag, and Lehec obtain the required bounds with universal constants, using stochastic localization and an auxiliary construction in a higher-dimensional space.

There is an unusual aspect to how this proof was found. Bizeul, Klartag, and Lehec state that ChatGPT found most of the proofs and mathematical ideas in their paper. They attribute the auxiliary construction to themselves and describe their role as understanding the proofs and improving their exposition (Bizeul et al. 2026).

The geometric statement also has algorithmic consequences: it controls how quickly random walks explore convex bodies, and hence affects sampling and volume computation. I discussed the latter problem in an earlier post. After every linear direction has variance 1, the fluctuations of every slowly varying measurement are controlled on a scale independent of dimension.

References

- Bizeul, Pierre, Boaz Klartag, and Joseph Lehec. 2026. *Presenting a Proof of the Kannan–Lovász–Simonovits Conjecture*. <https://arxiv.org/abs/2610.05474>.
- Eldan, Ronen. 2013. “Thin Shell Implies Spectral Gap up to Polylog via a Stochastic Localization Scheme.” *Geometric and Functional Analysis* 23 (2): 532–69. <https://doi.org/10.1007/s00039-013-0214-y>.
- Eldan, Ronen, and Boaz Klartag. 2011. “Approximately Gaussian Marginals and the Hyperplane Conjecture.” In *Concentration, Functional Inequalities and Isoperimetry*, vol. 545. Contemporary Mathematics. American Mathematical Society. <https://doi.org/10.1090/conm/545/10764>.
- Kannan, Ravi, László Lovász, and Miklós Simonovits. 1995. “Isoperimetric Problems for Convex Bodies and a Localization Lemma.” *Discrete & Computational Geometry* 13 (3–4): 541–59. <https://doi.org/10.1007/BF02574061>.
- Klartag, Boaz, and Joseph Lehec. 2025a. “Affirmative Resolution of Bourgain’s Slicing Problem Using Guan’s Bound.” *Geometric and Functional Analysis* 35 (4): 1147–68. <https://doi.org/10.1007/s00039-025-00718-w>.

- Klartag, Boaz, and Joseph Lehec. 2025b. *Thin-Shell Bounds via Parallel Coupling*. <https://arxiv.org/abs/2507.15495>.
- Letwin, Brayden. 2026. *The KLS Constant Is $O(\log^{1/4} n)$* . <https://arxiv.org/abs/2607.24164>.
- Milman, Emanuel. 2009. “On the Role of Convexity in Isoperimetry, Spectral Gap and Concentration.” *Inventiones Mathematicae* 177 (1): 1–43. <https://doi.org/10.1007/s00222-009-0175-9>.
- Song, Zhao, and Xinzhi Zhang. 2026a. *An $O(1)$ Bound for the KLS Constant*. <https://arxiv.org/abs/2610.01447v2>.
- Song, Zhao, and Xinzhi Zhang. 2026b. *An $O(4^{\log^* n})$ Bound for the KLS Constant*. <https://arxiv.org/abs/2610.01447v1>.