

Almost balls have almost minimal perimeter

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This post continues my series on favorite theorems of the twenty-first century. For an overview of the categories and earlier selections, see my introductory post.

My choice for 2005 in Geometry and Topology is the sharp quantitative isoperimetric inequality of Nicola Fusco, Francesco Maggi and Aldo Pratelli. Their paper was received by the *Annals of Mathematics* in December 2005 and eventually published in 2008 (Fusco et al. 2008).

The classical *isoperimetric inequality* says that, among all bodies in $\mathbb{R}^n$ with a given volume, the ball has the smallest perimeter. In dimension 2, this is the familiar statement that among all plane figures of a given area, the circle has the shortest boundary.

More precisely, let $E \subset \mathbb{R}^n$ be a Borel set with

$$0 < |E| < \infty,$$

where $|E|$ denotes its n -dimensional volume. Let $P(E)$ denote its perimeter. For a set with a smooth boundary, this is simply the $(n - 1)$ -dimensional area of its boundary.

If B is any ball with

$$|B| = |E|,$$

then the isoperimetric inequality states that

$$P(E) \geq P(B),$$

with equality precisely when E itself is a ball, up to changing a set of measure zero.

This tells us exactly which sets minimize perimeter, but it raises a natural stability question:

If $P(E)$ is only slightly larger than $P(B)$, must E be close to a ball?

To make this precise, define the *isoperimetric deficit*

$$D(E) = \frac{P(E) - P(B)}{P(B)}.$$

Thus

$$D(E) \geq 0,$$

and $D(E) = 0$ exactly for balls. A small value of $D(E)$ means that E is almost optimal in the isoperimetric inequality.

We also need to measure how far E is from being a ball. If X and Y are two sets, their *symmetric difference* is

$$X \triangle Y = (X \setminus Y) \cup (Y \setminus X).$$

Its volume $|X \triangle Y|$ measures how much of the two sets fails to overlap.

Let $\mathcal{B}$ be the collection of all balls B with $|B| = |E|$. The *Fraenkel asymmetry* of E is

$$\lambda(E) = \min_{B \in \mathcal{B}} \frac{|E \triangle B|}{r(B)^n},$$

where $r(B)$ denotes the radius of B . Thus $\lambda(E)$ is small precisely when, after choosing the best possible center, E differs little in volume from a ball.

The question is therefore whether small $D(E)$ forces small $\lambda(E)$, and how strong this relation can be.

In 1992, Robert Hall proved that for every dimension n there is a constant C_n such that

$$\lambda(E) \leq C_n D(E)^{1/4}$$

(Hall 1992). He conjectured that the exponent $1/4$ could be replaced by $1/2$.

Fusco, Maggi and Pratelli proved exactly this.

Theorem 1 *For every integer $n \geq 2$ there exists a constant C_n such that, for every Borel set $E \subset \mathbb{R}^n$ with $0 < |E| < \infty$,*

$$\lambda(E) \leq C_n \sqrt{D(E)}.$$

Equivalently, one can write the conclusion as

$$D(E) \geq c_n \lambda(E)^2$$

for some constant $c_n > 0$. In this form the meaning is especially clear: if a set is noticeably different from every ball, then its perimeter must exceed the optimal perimeter by an amount at least proportional to the square of that difference.

The exponent 2 in this last inequality, or equivalently the exponent $1/2$ in the theorem, is best possible.

To see why, consider an ellipsoid which is only slightly deformed from a ball. For example, stretch a ball by a factor $1 + \epsilon$ in one direction and rescale slightly in the other directions so that its volume stays fixed. When ϵ is small, its Fraenkel asymmetry is of order

$$\epsilon,$$

while its excess perimeter is only of order

$$\epsilon^2.$$

Thus

$$\lambda(E) \asymp \sqrt{D(E)}$$

for such nearly spherical ellipsoids. No estimate with a power of $D(E)$ larger than $1/2$ can therefore hold for all sets.

This is one reason the theorem is particularly satisfying: it does not merely prove that an almost optimal set is almost a ball, but gives the best possible dependence between the two notions of “almost”.

I particularly like this theorem because it strengthens one of the most classical statements in geometry in exactly the right way. The ordinary isoperimetric inequality says

$$D(E) = 0 \implies E \text{ is a ball.}$$

The quantitative theorem says much more:

$$D(E) \text{ small} \implies E \text{ is quantitatively close to a ball,}$$

and it gives the optimal rate

$$\lambda(E) = O(\sqrt{D(E)}).$$

This sharp stability principle has since become a model for many other quantitative geometric inequalities. Instead of asking only for the exact optimizers of an inequality, one asks whether every near-optimizer must be close to an optimizer, and with what best possible rate. The theorem of Fusco, Maggi and Pratelli is one of the cleanest examples of this philosophy.

References

- Fusco, Nicola, Francesco Maggi, and Aldo Pratelli. 2008. “The Sharp Quantitative Isoperimetric Inequality.” *Annals of Mathematics* 168 (3): 941–80. <https://doi.org/10.4007/annals.2008.168.941>.
- Hall, Robert R. 1992. “A Quantitative Isoperimetric Inequality in n -Dimensional Space.” *Journal für Die Reine Und Angewandte Mathematik* 428: 161–76.