Zeta(5) is irrational

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On September 17, 2026, Aabir Fauzan posted a preprint entitled $\zeta(5)$ *is irrational* (Fauzan 2026). If correct, its main result settles one of the best-known open problems about special values of the Riemann zeta function.

For a real number $s > 1$, the *Riemann zeta function* is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Thus

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots, \qquad \zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots,$$

and

$$\zeta(5) = 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \cdots = 1.036927\ldots.$$

Euler discovered that all values at positive even integers can be expressed in terms of powers of π. For example,

$$\zeta(2) = \frac{\pi^2}{6}, \qquad \zeta(4) = \frac{\pi^4}{90}.$$

More generally, $\zeta(2n)$ is a nonzero rational multiple of π^{2n}. Since π is transcendental, all these numbers are transcendental as well.

For odd integers the situation is completely different. No comparable formula is known, and even very basic arithmetic questions have proved extraordinarily difficult.

In 1978, Roger Apéry famously proved that

$$\zeta(3)$$

is irrational (Apéry 1979). His theorem was a major surprise. Yet for almost half a century afterwards, no other particular odd zeta value

$$\zeta(5),\ \zeta(7),\ \zeta(9),\ldots$$

was known to be irrational.

There was nevertheless substantial progress if several zeta values were considered simultaneously. Around 2000, Rivoal and Ball–Rivoal proved that infinitely many of the numbers

$$\zeta(3), \zeta(5), \zeta(7), \ldots$$

are irrational (Rivoal 2000; Ball and Rivoal 2001). Zudilin proved that at least one of

$$\zeta(5), \quad \zeta(7), \quad \zeta(9), \quad \zeta(11)$$

is irrational (Zudilin 2001). Later results gave increasingly strong lower bounds for how many odd zeta values must be irrational.

These theorems do not, however, tell us which ones are irrational. Before the new work, $\zeta(3)$ remained the only specified odd zeta value for which irrationality had been established.

The new result is therefore particularly striking.

Theorem 1 *The number*

$$\zeta(5) = \sum_{n=1}^{\infty} \frac{1}{n^5}$$

is irrational.

The final irrationality argument is remarkably simple. Fauzan constructs, for every sufficiently large positive integer n, a polynomial

$$Q_n(X) \in \mathbf{Z}[X]$$

of degree $37n$ such that

$$0 < Q_n(\zeta(5)) < \exp\left(-\frac{139}{5}n^2\right).$$

Indeed, suppose for contradiction that

$$\zeta(5) = \frac{a}{b}$$

for some integers a and $b > 0$. Since Q_n has integer coefficients and degree $37n$,

$$b^{37n} Q_n(a/b)$$

is a positive integer. But

$$0 < b^{37n} Q_n(a/b) < \exp\left(37n \log b - \frac{139}{5}n^2\right),$$

and the right-hand side tends to zero as $n \to \infty$. For sufficiently large n this gives a positive integer smaller than 1, a contradiction.

Thus all the work lies in constructing the polynomials Q_n.

They come from specially chosen Hankel determinants whose entries depend linearly on the polynomial variable. At $X = \zeta(5)$, the matrix becomes a positive moment matrix, so its determinant is strictly positive. A multiple-integral representation then gives a strong upper bound for this determinant using logarithmic-energy estimates.

The remaining difficulty is arithmetic: the determinant initially has rational coefficients. Fauzan estimates its denominators prime by prime and shows that, after normalization to an integer polynomial, the cost of clearing the denominators is still smaller than the analytic decay. This produces the required polynomials Q_n.

The construction also gives more than irrationality. Fauzan proves that, for every sufficiently large positive integer b and every integer a,

$$\left|\zeta(5) - \frac{a}{b}\right| > b^{-260}.$$

Thus the irrationality exponent of $\zeta(5)$ is at most 260. The constant is certainly not expected to be optimal; what is important is that the method gives an explicit finite bound.

There had already been impressive progress towards approximating $\zeta(5)$. In recent work, Francis Brown and Wadim Zudilin constructed infinitely many effective rational approximations p/q satisfying

$$0 < \left|\zeta(5) - \frac{p}{q}\right| < q^{-0.86}$$

(Brown and Zudilin 2026). This is not enough by itself to prove irrationality. Fauzan's approach instead uses polynomials of increasing degree whose values at $\zeta(5)$ decay quadratically exponentially in the degree parameter.

Because the result would be the first proof of irrationality of an individual odd zeta value since Apéry's theorem, the preprint immediately attracted considerable scrutiny. One particularly notable development is that a complete formalization of the proof in Lean 4 has since been produced under the direction of Dan Romik (Romik 2026). The formal development contains no unproved "sorry" statements; apart from the standard logical axioms used by Mathlib, its only external mathematical input is the Prime Number Theorem, which is inserted in its standard form because the required version is not presently contained in Mathlib.

The formalization also exposed a few places where the paper's terse arguments needed to be expanded. In particular, one inequality used in the local estimates was stated without proof in the preprint; the formalization supplies a proof. Several other steps were formalized using slightly different arguments. Thus the formal verification is not merely a mechanical transcription but also provides a detailed independent audit of the construction.

The result, if it survives the usual process of mathematical scrutiny, would mark an extraordinary step in a problem that has been almost stationary at the level of individual odd zeta values since 1978. We have long known that infinitely many odd zeta values are irrational, but until now we could point to only one of them by name:

$$\zeta(3).$$

The new theorem adds the next one:

$$\zeta(5) \text{ is irrational.}$$

References

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