Small triangulations of projective space

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On September 9, 2026, Florian Frick, Kaave Hosseini, Eric Myzelev, Arya Narnapatti and Aliaksei Vasileuski posted a preprint (Frick et al. 2026) which gives a superpolynomial lower bound for the number of vertices needed to triangulate real projective space. Together with constructions found only a few years ago, their theorem essentially determines the asymptotic answer to a long-standing problem in combinatorial topology.

Recall that the d-dimensional *real projective space* $\mathbb{RP}^d$ is the set of all lines through the origin in $\mathbb{R}^{d+1}$. Equivalently, consider the unit sphere

$$S^d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}.$$

Every line through the origin meets S^d in two opposite points x and $-x$, so $\mathbb{RP}^d$ can be obtained from S^d by identifying every point x with its antipodal point $-x$.

For example, $\mathbb{RP}^1$ is homeomorphic to a circle. The space $\mathbb{RP}^2$ is the usual real projective plane.

A finite *simplicial complex* consists of a finite set of vertices and a collection of finite subsets of these vertices, called *simplices*, such that every subset of a simplex is again a simplex. A simplex with two vertices is an edge, one with three vertices is a triangle, one with four vertices is a tetrahedron, and so on. By gluing geometric simplices according to this combinatorial information, one obtains the *geometric realization* of the complex.

A *simplicial triangulation* of a topological space M is a finite simplicial complex whose geometric realization is homeomorphic to M. Thus, informally, the problem is to build M by gluing together simplices.

A natural measure of the complexity of such a triangulation is its number of vertices. Let μ_d denote the smallest possible number of vertices in a simplicial triangulation of $\mathbb{RP}^d$.

How quickly does μ_d grow with d?

There is an elementary-looking construction with exponentially many vertices. Start with a suitable antipodally symmetric triangulation of S^d and identify opposite points. In particular, a classical construction gives

$$\mu_d \leq 2^{d+1} - 1$$

up to an inessential shift of the dimension.

For a long time no construction with substantially fewer vertices was known. On the other hand, the best general lower bounds were only polynomial. In particular, Arnoux and Marin proved that, for $d \geq 3$,

$$\mu_d \geq \binom{d+2}{2} + 1$$

(Arnoux and Marin 1991).

Thus there was an enormous gap: perhaps the correct answer was polynomial in d, perhaps exponential, or perhaps somewhere in between.

A breakthrough came from Karim Adiprasito, Sergey Avvakumov and Roman Karasev. In a paper published in 2022, they constructed triangulations of $\mathbb{RP}^d$ with

$$\exp\left(\left(\frac{1}{2} + o(1)\right)\sqrt{d}\log d\right)$$

vertices (Adiprasito et al. 2022). This was the first construction to break the exponential barrier.

Their argument was surprisingly short. They first constructed a centrally symmetric polytope whose boundary has an antipodal symmetry with special combinatorial properties, and then took its quotient by this symmetry to obtain a triangulation of projective space.

The combinatorial part of their construction was subsequently improved by Peter Frankl, J'anos Pach and D"om"ot"or P'alv"olgyi (Frankl et al. 2022). Combining the two works gives

$$\mu_d \leq \exp\Big(O(\sqrt{d\log d})\Big).$$

This showed that the answer is subexponential. But it left open the opposite question: could there be dramatically smaller triangulations, perhaps even ones with only polynomially many vertices?

The new theorem rules this out.

Theorem 1 *Every finite simplicial triangulation of $\mathbb{RP}^d$, $d \geq 1$, has at least*

$$\frac{1}{2}\exp\left(\sqrt{\frac{d}{24e^2}}\right)$$

vertices. In particular,

$$\mu_d = \exp(\Omega(\sqrt{d})).$$

Here the notation

$$\exp(\Omega(\sqrt{d}))$$

means that there is an absolute constant $c > 0$ such that, for all sufficiently large d,

$$\mu_d \geq \exp(c\sqrt{d}).$$

Together with the known upper bound, we therefore have

$$\exp(\Omega(\sqrt{d})) \leq \mu_d \leq \exp\Big(O(\sqrt{d\log d})\Big).$$

Since

$$\sqrt{\log d} = d^{o(1)},$$

this can be summarized by the remarkably precise formula

$$\boxed{\mu_d = \exp\Big(d^{1/2+o(1)}\Big).}$$

Thus the new theorem determines the correct power of d in the exponent. There remains a factor of order $\sqrt{\log d}$ between the best known upper and lower bounds for $\log \mu_d$, but the main asymptotic question is settled: the minimum number of vertices is much larger than every polynomial in d, yet much smaller than $\exp(cd)$.

References

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