Automatic numbers in independent bases

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Two papers by Boris Adamczewski and Colin Faverjon have just been published back-to-back in the *Annals of Mathematics*: the paper *Mahler's method in several variables and finite automata* (Adamczewski and Faverjon 2026b), followed immediately by an addendum (Adamczewski and Faverjon 2026a). Among their consequences is a striking theorem about numbers whose digits can be generated by finite automata.

Let

$$\mathbf{u} = (u_0, u_1, u_2, \ldots)$$

be an infinite sequence, and let $k \geq 2$ be an integer. The k*-kernel* of $\mathbf{u}$ is the collection of all sequences

$$(u_{k^i n+j})_{n\geq 0}\,, \qquad i \geq 0, \quad 0 \leq j < k^i.$$

The sequence $\mathbf{u}$ is called k*-automatic* if its k-kernel is finite (Allouche and Shallit 2003).

There are many equivalent definitions. Perhaps the most intuitive one is that a k-automatic sequence can be generated by a finite automaton: given the digits of n written in base k, the automaton outputs u_n. A finite automaton has only finitely many possible internal states, so it can be thought of as a very simple computer with no unbounded memory.

A standard example is the *Thue–Morse sequence*

$$0, 1, 1, 0, 1, 0, 0, 1, \ldots.$$

Its n-th term t_n is 0 if the binary expansion of n contains an even number of 1's, and 1 otherwise. It satisfies

$$t_{2n} = t_n, \qquad t_{2n+1} = 1 - t_n.$$

It follows that its 2-kernel contains only the Thue–Morse sequence itself and its complement, so the sequence is 2-automatic.

We can use automatic sequences to define real numbers. Let $b \geq 2$. A real number x is called *automatic in base* b if the sequence of digits in its base-b expansion is k-automatic for some integer $k \geq 2$. Notice that b and k need not be the same: b is the base in which the number itself is written, while k is the base in which the automaton reads the index of a digit.

For example, the binary Thue–Morse number

$$\tau = \sum_{n=0}^{\infty} \frac{t_n}{2^{n+1}} = 0.0110100110010110\ldots_2$$

is automatic in base 2.

Automatic numbers have extremely structured digit expansions. This makes them very different from what one expects from irrational algebraic numbers. In 2007, Adamczewski and Bugeaud proved that an irrational algebraic number cannot have an automatic expansion (Adamczewski and Bugeaud 2007). Equivalently, every irrational automatic real number is transcendental.

The new result goes much further by comparing expansions in different bases.

Two integers $b, b' \geq 2$ are called *multiplicatively independent* if

$$\frac{\log b}{\log b'}$$

is irrational. Equivalently, there are no positive integers m, n such that

$$b^m = (b')^n.$$

For example, 2 and 10 are multiplicatively independent, while 4 and 8 are not, since

$$4^3 = 8^2.$$

Finally, recall that $\overline{\mathbf{Q}}$ denotes the field of algebraic numbers. Real numbers $x_1, \ldots, x_r$ are called *algebraically independent over* $\overline{\mathbf{Q}}$ if there is no nonzero polynomial

$$P(X_1, \ldots, X_r)$$

with algebraic coefficients such that

$$P(x_1, \ldots, x_r) = 0.$$

Adamczewski and Faverjon prove the following.

Theorem 1 *Let $r \geq 1$, and let $b_1, \ldots, b_r \geq 2$ be pairwise multiplicatively independent integers. For every i, let x_i be a real number which is automatic in base b_i. If none of the numbers $x_1, \ldots, x_r$ is rational, then they are algebraically independent over $\overline{\mathbf{Q}}$.*

Even the simplest consequences are remarkable.

For $r = 1$, algebraic independence just means transcendence, so the theorem contains the result that every irrational automatic number is transcendental.

Now take $r = 2$ and suppose that the same real number x is automatic in two multiplicatively independent bases b_1 and b_2. The two numbers

$$x_1 = x, \qquad x_2 = x$$

are certainly algebraically dependent, since

$$x_1 - x_2 = 0.$$

The theorem therefore implies that x must be rational.

Thus an irrational real number cannot be automatic in two multiplicatively independent bases.

For example, the Thue–Morse number τ is automatic in base 2. Since 2 and 10 are multiplicatively independent and τ is irrational, the decimal expansion of τ cannot be automatic. In other words, the strikingly simple pattern visible in its binary expansion must disappear when the same number is written in decimal.

The full theorem says much more than this. Suppose that x_1 is an irrational automatic number in base 2, x_2 is one in base 3, and x_3 is one in base 5. Then there cannot be *any* algebraic relation between them: not only can none of them be algebraic, but we cannot have, for example,

$$x_3 = x_1 + x_2, \qquad x_1 x_2 = x_3^2,$$

or any other polynomial identity with algebraic coefficients.

There is a particularly concrete family of examples. Using the Thue–Morse digits t_n, define

$$\tau_b = \sum_{n=0}^{\infty} \frac{t_n}{b^{n+1}}, \qquad b \geq 2.$$

The number τ_b is automatic in base b, since its base-b digits form the same 2-automatic Thue–Morse sequence. Hence, whenever $b_1, \ldots, b_r$ are pairwise multiplicatively independent, the numbers

$$\tau_{b_1}, \ldots, \tau_{b_r}$$

are algebraically independent.

There is a subtle point in the phrase “pairwise multiplicatively independent”. For two bases there is no distinction, but for three or more bases pairwise independence is weaker than simultaneous multiplicative independence. For example,

$$2, \quad 3, \quad 6$$

are pairwise multiplicatively independent: no power of one is a power of another. Nevertheless, they satisfy the multiplicative relation

$$2 \cdot 3 = 6.$$

The main paper (Adamczewski and Faverjon 2026b) established the corresponding automatic-number theorem under the stronger simultaneous independence assumption. The short addendum (Adamczewski and Faverjon 2026a) removes this restriction and gives the pairwise version stated above. Thus the theorem applies, for example, simultaneously to bases 2, 3, and 6.

The starting point of the proof is an observation made by Alan Cobham in 1968 (Cobham 1968). Suppose that $(u_n)_{n\geq 0}$ is a k-automatic sequence and form its generating function

$$f(z) = \sum_{n=0}^{\infty} u_n z^n.$$

Then f satisfies a functional equation of the form

$$p_0(z)f(z) + p_1(z)f(z^k) + \cdots + p_m(z)f(z^{k^m}) = 0,$$

where $p_0, \ldots, p_m$ are polynomials, not all zero. Functions satisfying equations of this type are called *Mahler functions*.

This gives a bridge between finite automata and transcendence theory. If

$$x = 0.a_1a_2a_3\ldots_b$$

is automatic in base b, its digits give a Mahler function, and x is obtained by evaluating this function at the algebraic point $1/b$. Questions about simple digit expansions can therefore be transformed into questions about algebraic relations between values of Mahler functions.

The main achievement of Adamczewski and Faverjon is a far-reaching theory for algebraic independence of such values in several variables. The general results occupy most of their Annals paper. The automatic-number theorem is then obtained by applying this theory at the points

$$\frac{1}{b_1}, \ldots, \frac{1}{b_r}.$$

The addendum introduces an additional argument, originating in work of Loxton and van der Poorten, which allows the stronger pairwise independence hypothesis and leads to the theorem above.

It is commonly expected that expansions in multiplicatively independent bases should have essentially unrelated structure. If an irrational number looks exceptionally simple in base 2, there should be no reason for it to remain simple in base 10. Yet even very weak versions of this intuition are notoriously difficult to prove: we know remarkably little about the digits of familiar irrational algebraic numbers.

For automatic expansions, Adamczewski and Faverjon obtain an especially strong form of this principle. Simple expansions in independent bases do not merely have to be different. Unless a rational number intervenes, the numbers represented by them cannot satisfy even a single algebraic relation.

References

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