The Ten Martini Problem

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This post continues my series on favorite theorems of the twenty-first century. For an overview of the categories and earlier selections, see this post.

My choice for 2005 in Analysis is Artur Avila and Svetlana Jitomirskaya's solution of the Ten Martini Problem. Their preprint was posted in March 2005 and eventually published in the *Annals of Mathematics* in 2009 (Avila and Jitomirskaya 2009).

The theorem concerns a particularly simple operator acting on infinite sequences. Let $\ell^2(\mathbb{Z})$ be the set of all doubly infinite sequences

$$x = (\ldots, x_{-1}, x_0, x_1, \ldots)$$

of complex numbers such that

$$\sum_{n\in\mathbb{Z}} |x_n|^2 < \infty.$$

For real numbers λ, α, θ, with $\lambda \neq 0$, the *almost Mathieu operator*

$$H_{\lambda,\alpha,\theta} : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$$

is defined by

$$(H_{\lambda,\alpha,\theta}x)_n = x_{n+1} + x_{n-1} + 2\lambda\cos\Big(2\pi(\theta + n\alpha)\Big)x_n, \qquad n \in \mathbb{Z}.$$

The parameters λ, α, and θ are called the *coupling*, *frequency*, and *phase*, respectively.

The formula looks elementary: to obtain the n-th term of Hx, we add the two neighbouring terms x_{n-1} and x_{n+1} and a multiple of x_n. The difficulty comes from the coefficient

$$2\lambda\cos\Big(2\pi(\theta + n\alpha)\Big).$$

If α is rational, this coefficient is periodic in n. If α is irrational, it never repeats periodically, although it repeatedly comes arbitrarily close to its previous values. Such operators are called *quasiperiodic*.

The almost Mathieu operator arose in mathematical physics as a model for the motion of an electron in a two-dimensional crystal subject to a magnetic field. It is also closely connected with the famous Hofstadter butterfly, the complicated fractal-looking diagram obtained by plotting the possible energy levels against the magnetic flux.

The central object of study is the *spectrum* of $H_{\lambda,\alpha,\theta}$. For a real number E, let I denote the identity operator and consider

$$H_{\lambda,\alpha,\theta} - EI.$$

The spectrum $\sigma(H_{\lambda,\alpha,\theta})$ is the set of all $E \in \mathbb{R}$ for which this operator is not invertible. Informally, the spectrum is the set of energy values permitted by the system.

For irrational α, the spectrum does not depend on the phase θ. We may therefore write it as

$$\Sigma_{\lambda,\alpha}.$$

If $\alpha = p/q$ is rational, in lowest terms, the spectrum is a union of q closed intervals, called *bands*, some of which may touch at their endpoints. The picture for irrational α is dramatically different.

Recall that a *Cantor set* is a nonempty compact subset of $\mathbb{R}$ which is perfect and totally disconnected. Perfect means that it has no isolated points: every point of the set is approached by other points of the set. Totally disconnected means that the only connected subsets are individual points. The standard middle-thirds Cantor set is the most familiar example.

Thus a Cantor set consists of infinitely many pieces separated by gaps, with new gaps appearing at arbitrarily small scales. It can contain uncountably many points while containing no interval at all.

Already in 1964, work of Azbel suggested that the spectrum in the irrational case should have this Cantor structure (Azbel 1964). The problem later acquired one of the most memorable names in mathematics.

At the 1981 annual meeting of the American Mathematical Society, Mark Kac asked whether “all gaps are there” and offered ten martinis for a solution. Barry Simon subsequently popularized two versions of the question (Simon 1982). The Ten Martini Problem asks whether the spectrum of the almost Mathieu operator is always a Cantor set. The stronger Dry Ten Martini Problem asks whether every spectral gap allowed by the general gap-labelling theory is actually open.

Substantial progress followed. Bellissard and Simon proved Cantor spectrum for a topologically generic set of parameters (Bellissard and Simon 1982). Much later, Puig obtained the result for a set of parameters of full Lebesgue measure (Puig 2004). These theorems showed that Cantor spectrum was overwhelmingly common, but they did not rule out exceptional irrational frequencies.

Avila and Jitomirskaya eliminated all exceptions.

Theorem 1 *Let $\lambda \neq 0$, let α be irrational, and let $\theta \in \mathbb{R}$. Then the spectrum of the almost Mathieu operator*

$$(H_{\lambda,\alpha,\theta}x)_n = x_{n+1} + x_{n-1} + 2\lambda \cos\Big(2\pi(\theta + n\alpha)\Big)x_n$$

is a Cantor set.

What I especially like about this theorem is the contrast between the simplicity of the operator and the complexity of its spectrum. The definition uses only addition and cosine, yet for every irrational frequency, with no exceptions whatsoever, the allowed energies form a fractal set with gaps at every scale.

The phrase “every irrational frequency” is crucial. Irrational numbers can have very different arithmetic properties. Some, called *Diophantine*, cannot be approximated unusually well by rational numbers. Others, called *Liouville numbers*, have extraordinarily good rational approximations. Techniques that work well in one regime can fail completely in the other.

This arithmetic difficulty is one reason why the Ten Martini Problem resisted a complete solution for so long. Before Avila and Jitomirskaya, the known results already covered parameter sets which were both topologically large and of full Lebesgue measure. What remained was a thin exceptional set lying between regimes governed by very different methods.

A second remarkable feature is a duality between small and large coupling. The spectra satisfy

$$\Sigma_{\lambda,\alpha} = \lambda \Sigma_{\lambda^{-1},\alpha}$$

for $\lambda > 0$. Thus the regimes $0 < \lambda < 1$ and $\lambda > 1$ reflect one another. The value

$$|\lambda| = 1$$

is the *critical* case separating them and has particularly rich behaviour.

Avila and Jitomirskaya's proof brings together ideas from spectral theory and dynamical systems, including localization, reducibility, Lyapunov exponents, rational approximation and analytic continuation. Roughly speaking, different arithmetic types of α require different approaches, and one of the achievements of the proof is to make these approaches meet so that no irrational frequency is left uncovered.

There is another beautiful question about the same spectrum: not its topology, but its size. Aubry and Andr'e conjectured that its Lebesgue measure, that is, its total length, is

$$4\Big|1 - |\lambda|\Big|$$

for every irrational α (Aubry and André 1980). Jitomirskaya and Krasovsky proved this when $|\lambda| \neq 1$ (Jitomirskaya and Krasovsky 2002). Last had earlier proved the critical case for almost every irrational frequency (Last 1994), and Avila and Krikorian completed it for all irrational frequencies (Avila and Krikorian 2006).

In particular, when $|\lambda| = 1$, the spectrum has Lebesgue measure zero. Combining this with the Ten Martini theorem gives a striking picture: the critical almost Mathieu operator has an uncountable spectrum, with no isolated points, but its total length is zero.

Even Lebesgue measure does not fully describe how thin such a set is. A finer notion is the *Hausdorff dimension*. An interval has Hausdorff dimension 1, while a finite or countable set has Hausdorff dimension 0; Cantor sets may have any intermediate dimension. In 2019, Jitomirskaya and Krasovsky proved that for the critical almost Mathieu operator and every irrational α, the Hausdorff dimension of the spectrum is at most

$$\frac{1}{2}$$

(Jitomirskaya and Krasovsky 2019).

The stronger Dry Ten Martini Problem has also seen major progress. In 2023, Avila, Jiangong You and Qi Zhou proved that all gaps predicted by gap labelling are open whenever $|\lambda| \neq 1$ (Avila et al. 2023). The full problem at the critical value $|\lambda| = 1$ remains open.

For me, the Ten Martini theorem is a particularly beautiful result in analysis because it combines an elementary formula, a striking geometric conclusion, a connection with quantum physics, and a genuinely difficult interaction between analysis, dynamics and number theory. Starting from the very concrete rule

$$x_n \longmapsto x_{n+1} + x_{n-1} + 2\lambda \cos\Big(2\pi(\theta + n\alpha)\Big)x_n,$$

one arrives inevitably, for every irrational α, at a Cantor set.

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