Banach’s isometric conjecture is true

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In August 2026, Xinbao Lu and Kaiwen Yang posted a preprint (Lu and Yang 2026) that completed the real case of a problem posed by Stefan Banach in 1932 (Banach 1932). The question has a remarkably elementary formulation: if all subspaces of one fixed dimension inside a normed space look exactly the same, must the entire norm come from Euclidean geometry?

Let X be a real vector space. A *norm* on X is a function $\|\cdot\| : X \to [0,\infty)$ such that $\|x\| = 0$ only for $x = 0$,

$$\|\lambda x\| = |\lambda|\,\|x\|, \qquad \|x+y\| \le \|x\| + \|y\|$$

for all $x, y \in X$ and $\lambda \in \mathbb{R}$. A normed space is called a *Banach space* if every Cauchy sequence converges. In finite dimensions this completeness condition is automatic.

An *inner product* on X is a symmetric bilinear function $\langle\cdot,\cdot\rangle$ such that $\langle x, x\rangle > 0$ for every $x \neq 0$. It defines the norm

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

A Banach space whose norm is induced by an inner product is called a *Hilbert space*. Thus finite-dimensional Hilbert spaces are precisely Euclidean spaces after a linear change of coordinates.

Two linear subspaces $E, F \subset X$ are called *linearly isometric* if there is a linear bijection $T : E \to F$ such that

$$\|Tx\| = \|x\| \qquad \text{for all } x \in E.$$

In a Hilbert space, any two subspaces of the same finite dimension are linearly isometric. Banach asked whether the converse is true: if, for some fixed integer n with $2 \le n < \dim X$, all n-dimensional subspaces of X are linearly isometric, must X be a Hilbert space?

Theorem (Banach's isometric conjecture, real case). Let X be a real Banach space. Suppose that, for some fixed integer $2 \le n < \dim X$, all n-dimensional linear subspaces of X are linearly isometric. Then X is a Hilbert space.

For odd n, this is the new theorem of Lu and Yang (Lu and Yang 2026); for even n, it was proved by Gromov in 1967 (Gromov 1967). Together these results settle Banach's conjecture over the real numbers.

There is an equivalent and very geometric way to state the finite-dimensional problem. The *unit ball* of a normed space is

$$B_X = \{x \in X : \|x\| \le 1\}.$$

If X is finite-dimensional, then B_X is a convex body: it is compact, convex, and has nonempty interior. It is also *origin-symmetric*, meaning that $x \in B_X$ if and only if $-x \in B_X$.

If $H \subset X$ is a linear subspace, then $B_X \cap H$ is exactly the unit ball of the restricted norm on H. Such intersections are called *central sections*, since the subspace H passes through the origin. Hence two subspaces are linearly isometric precisely when their corresponding sections of B_X are linearly equivalent. Here two sets A and B lying in vector spaces of the same dimension are *linearly equivalent* if some invertible linear map sends A onto B.

An *ellipsoid centred at the origin* is the image of a Euclidean unit ball under an invertible linear map. Therefore a finite-dimensional norm comes from an inner product if and only if its unit ball is an ellipsoid. Banach's conjecture is consequently equivalent to the following statement.

Convex-geometric form. Let $K \subset \mathbb{R}^N$ be an origin-symmetric convex body, and let $2 \le n < N$. If all intersections $K \cap H$, where H runs over the n-dimensional linear subspaces of $\mathbb{R}^N$, are linearly equivalent, then K is an ellipsoid.

The origin-symmetry assumption is important in this formulation, but it is automatic for unit balls. The new part of Lu and Yang's work is the case where n is odd. Moreover, it is enough to solve the codimension-one case $N = n + 1$. Indeed, once every $(n+1)$-dimensional space with the stated property is known to be Hilbert, any two vectors x, y in a larger space lie in some $(n+1)$-dimensional subspace. The parallelogram identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

then holds for every pair x, y, which characterizes norms coming from inner products.

Thus the heart of the new result can be stated particularly simply.

Theorem (Lu–Yang). Let $n \geq 3$ be odd, and let $K \subset \mathbb{R}^{n+1}$ be an origin-symmetric convex body. If all central hyperplane sections of K are linearly equivalent, then K is an ellipsoid.

The history of the problem is unusually long. In 1935, Auerbach, Mazur and Ulam proved the real case $n = 2$ (Auerbach et al. 1935). Dvoretzky's theorem settled the infinite-dimensional real case (Dvoretzky 1959). Gromov then proved the conjecture for every even n, and also dealt with odd n whenever the ambient dimension is at least $n + 2$ (Gromov 1967). Thus the remaining finite-dimensional real problem was concentrated in odd dimension and codimension one.

In 2021, Bor, Hernández-Lamoneda, Jiménez-Desantiago and Montejano proved the conjecture for $n \equiv 1 \pmod 4$, $n \geq 5$, with the possible exception of $n = 133$ (Bor et al. 2021). In 2023, Ivanov, Mamaev and Nordskova settled the first genuinely difficult odd case, namely $n = 3$ and $\dim X = 4$ (Ivanov et al. 2023). Lu and Yang's theorem fills all the remaining odd-dimensional gaps.

Lu and Yang's theorem completed Banach's conjecture over the real numbers. Only a few days after their first preprint appeared, Acuaviva and Kania posted a paper completing the complex case as well, and also proving a quaternionic analogue (Acuaviva and Kania 2026). Their proof adapts the bundle-degree mechanism introduced by Lu and Yang. Thus, ninety-four years after Banach posed the question, the original real-or-complex isometric conjecture is now completely settled.

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