Quasiconformal uniformization of Sierpiński carpets

Bogdan Grechuk • 10 Sep 2026

https://functor.network/user/3333/entry/1975

A paper by Dimitrios Ntalampekos has recently been accepted for publication in *Inventiones Mathematicae* (Ntalampekos 2026). The paper gives a new quasiconformal uniformization theorem for subsets of the 2-sphere. In particular, it gives an exact criterion for when a Sierpiński carpet can be transformed by a quasiconformal homeomorphism into a round Sierpiński carpet.

We first introduce the objects appearing in the theorem. It is convenient to identify the 2-sphere with the Riemann sphere

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

We equip it with the usual spherical metric, that is, the great-circle distance after identifying $\widehat{\mathbb{C}}$ with the unit sphere in $\mathbb{R}^3$ by stereographic projection. For $a \in \widehat{\mathbb{C}}$ and $0 < r < \pi$, the open ball

$$B(a,r) = \{x \in \widehat{\mathbb{C}} : d(x,a) < r\}$$

is called a *geometric disk*. Its boundary is a circle on the sphere.

Recall that a Jordan curve is a subset homeomorphic to a circle. By the Jordan curve theorem, the complement of a Jordan curve in the sphere has exactly two connected components. Each of them is called a *Jordan region*.

A *Sierpiński carpet* is a compact set

$$S \subset \widehat{\mathbb{C}}$$

with empty interior such that the connected components

$$U_1, U_2, \ldots$$

of $\widehat{\mathbb{C}} \setminus S$ are Jordan regions whose closures are pairwise disjoint, and whose diameters tend to zero. More precisely, the last condition means that for every $\epsilon > 0$, only finitely many U_i satisfy

$$\operatorname{diam}(U_i) \geq \epsilon.$$

The regions U_i are called the *peripheral regions* of S.

A Sierpiński carpet is called *round* if all its peripheral regions are geometric disks.

The familiar standard Sierpiński carpet, obtained by repeatedly removing the middle square from a 3×3 subdivision of every remaining square, is therefore not round: its complementary regions are squares rather than disks. Topologically, however, the particular shapes of the holes make no difference. In fact, a classical theorem of Whyburn says that all Sierpiński carpets are homeomorphic. The question here is much more geometric: can one pass from a given carpet to a round one by a homeomorphism whose distortion is uniformly controlled?

This leads to quasiconformal maps. Let

$$f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

be a homeomorphism. For $x \in \widehat{\mathbb{C}}$ and $r > 0$, put

$$L_f(x, r) = \max_{d(x,y)=r} d(f(x), f(y))$$

and

$$l_f(x, r) = \min_{d(x,y)=r} d(f(x), f(y)).$$

Thus a small circle of radius r around x is mapped to a curve whose points have distances between $l_f(x, r)$ and $L_f(x, r)$ from $f(x)$.

For $K \geq 1$, we call f *K-quasiconformal* if

$$\limsup_{r \to 0} \frac{L_f(x, r)}{l_f(x, r)} \leq K$$

for every $x \in \widehat{\mathbb{C}}$. We call f *quasiconformal* if it is K-quasiconformal for some K. Thus a quasiconformal map may distort shapes, but its infinitesimal distortion is bounded uniformly over the sphere.

Two subsets $S, T \subset \widehat{\mathbb{C}}$ are called *quasiconformally equivalent* if there is a quasiconformal homeomorphism f of the sphere such that

$$f(S) = T.$$

We can now formulate the condition discovered by Ntalampekos. Let U and V be two disjoint Jordan regions. We say that the pair (U, V) is *K-quasiconformally circularizable* if there exists a K-quasiconformal homeomorphism

$$f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

such that both $f(U)$ and $f(V)$ are geometric disks.

A collection $\{U_i\}$ is called *uniformly quasiconformally pairwise circularizable* if there exists one constant $K \geq 1$ such that every pair U_i, U_j, $i \neq j$, is K-quasiconformally circularizable.

There is an important subtlety in this definition. We do *not* require one map to circularize all the U_i. For each pair U_i, U_j, we may use a completely different quasiconformal map. Only the bound K on the distortion has to be independent of the chosen pair.

Ntalampekos proved that this apparently pairwise condition is already enough to guarantee one global map which circularizes every peripheral region simultaneously.

Theorem. Let $S \subset \widehat{\mathbb{C}}$ be a Sierpiński carpet with peripheral regions $U_1, U_2, \ldots$. Then S is quasiconformally equivalent to a round Sierpiński carpet if and only if the family $\{U_i\}$ is uniformly quasiconformally pairwise circularizable.

One direction of the theorem is immediate. Suppose that a K-quasiconformal homeomorphism f transforms S into a round carpet. Then every $f(U_i)$ is a geometric disk. Consequently, the same map f circularizes every pair U_i, U_j, and the family is uniformly pairwise circularizable.

The converse is the remarkable part. We assume only that every pair of holes can be made round, with a uniform bound on the distortion, using maps which may depend on the pair. The theorem concludes that there is a *single* quasiconformal homeomorphism which makes all countably many holes round at once.

The result gives a sharp answer to a natural uniformization question for Sierpiński carpets, substantially extends the earlier theory, and at the same time forms part of a more general theorem covering Schottky sets and gaskets.

References

Ntalampekos, Dimitrios. 2026. “Quasiconformal Characterization of Schottky Sets.” *Inventiones Mathematicae*, ahead of print. https://doi.org/10.1007/s00222-026-01447-z.