# Finite-time blowup for the 3D Navier–Stokes equations

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On September 8, 2026, OpenAI announced a proposed solution of the Navier–Stokes Millennium Prize Problem (OpenAI 2026). The result goes in perhaps the more surprising of the two possible directions: rather than proving that every smooth three-dimensional fluid flow remains smooth forever, it constructs a flow which develops a singularity in finite time.

The Navier–Stokes equations describe the motion of an incompressible viscous fluid. Let

$$u = (u_1, u_2, u_3) : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$$

denote the velocity of the fluid, let

$$p : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$$

denote its pressure, and let

$$f = (f_1, f_2, f_3) : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$$

be an externally applied force.

For a constant $\nu > 0$, called the *viscosity*, the three-dimensional incompressible Navier–Stokes equations are

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i, \qquad i = 1, 2, 3,$$

together with

$$\mathrm{div}(u) = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0.$$

Here

$$\Delta u_i = \sum_{j=1}^{3} \frac{\partial^2 u_i}{\partial x_j^2}$$

is the Laplacian in the spatial variables.

The condition $\mathrm{div}(u) = 0$ expresses incompressibility. The term

$$\sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j}$$

describes the transport of the fluid by its own motion, while the term $\nu \Delta u_i$ describes the smoothing effect of viscosity.

An initial velocity

$$u^0 : \mathbb{R}^3 \to \mathbb{R}^3$$

is prescribed by

$$u(x, 0) = u^0(x).$$

A central question is whether smooth initial data can ever lead to a singularity. Informally, a singularity occurs if the velocity becomes unbounded after a finite amount of time. More precisely, if a smooth solution exists for $0 \leq t < T$, but

$$\limsup_{t \to T-} \sup_{x \in \mathbb{R}^3} |u(x, t)| = +\infty,$$

then the solution cannot be extended smoothly through the time $T$.

Why should such a singularity be difficult to produce? The nonlinear part of the equation can potentially concentrate the motion of the fluid, but viscosity acts in the opposite direction and tends to smooth the velocity field. The Millennium Prize Problem asks, in essence, which of these two effects ultimately wins in three dimensions.

There is also an important condition at spatial infinity. The official formulation (Fefferman 2000) requires the initial velocity and the external force to decrease very rapidly. For example, the force is required to satisfy inequalities of the form

$$|\partial_x^\alpha \partial_t^m f(x, t)| \leq C_{\alpha, m, K} (1 + |x| + t)^{-K}$$

for every multi-index $\alpha$, every non-negative integer $m$, and every $K > 0$, with a suitable constant $C_{\alpha, m, K}$.

Another requirement is bounded kinetic energy:

$$\int_{\mathbb{R}^3} |u(x, t)|^2 \, \mathrm{d}x \leq C$$

for a constant $C$ independent of time. Thus one is not allowed to obtain a singularity merely by letting the total energy become infinite.

The newly announced result can be formulated as follows.

**Theorem.** For every viscosity $\nu > 0$, there exists a smooth, rapidly decreasing external force

$$f : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$$

for which the three-dimensional incompressible Navier–Stokes equations have a smooth solution starting from rest,

$$u(x, 0) = 0,$$

which develops a singularity after a finite time. Throughout the time before the singularity, its kinetic energy remains uniformly bounded:

$$\sup_{0 \le t < T} \int_{\mathbb{R}^3} |u(x,t)|^2 \, \mathrm{d}x < +\infty,$$

but

$$\limsup_{t \to T-} \sup_{x \in \mathbb{R}^3} |u(x,t)| = +\infty.$$

There are two striking features in this statement. First, the fluid starts completely at rest. Thus no singular behavior has been hidden in the initial condition.

Second, the external force is smooth. One could trivially make a fluid move infinitely fast by applying an infinite force, but that would say nothing interesting about the Navier–Stokes equations. The theorem says that a perfectly regular force can initiate a flow whose own nonlinear dynamics eventually produces unbounded velocity.

How is it possible for the velocity to become arbitrarily large while

$$\int_{\mathbb{R}^3} |u|^2$$

stays bounded? There is no contradiction. A function may have an arbitrarily large maximum while being large only on an extremely small region. For a simple one-dimensional analogy, consider

$$g_n(x) = \begin{cases} n, & 0 \le x \le n^{-3}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\max_x |g_n(x)| = n \to \infty,$$

whereas

$$\int_{\mathbb{R}} |g_n(x)|^2 \, \mathrm{d}x = \frac{1}{n} \to 0.$$

The announced Navier–Stokes construction realizes a vastly more complicated version of this concentration phenomenon within an actual solution of the differential equations.

According to the description released with the proof (OpenAI 2026), the central object is a vortex: a spinning region of fluid which spirals inward while simultaneously becoming increasingly elongated. Its characteristic spatial scale shrinks while its velocity increases. In this way the maximum velocity can diverge even though the total energy remains finite.

There is another major obstacle. Suppose one first invented an arbitrary velocity field $u$ which blows up and then rearranged the Navier–Stokes equation to define

$$f_i = \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} - \nu \Delta u_i + \frac{\partial p}{\partial x_i}.$$

Usually the resulting force $f$ would itself become singular, which would not solve the problem.

The construction therefore requires a delicate cancellation. Near the singularity, several individual terms in the equation become very large: the acceleration, the nonlinear transport term, the pressure gradient, and the viscous term. They are arranged so that their leading large contributions cancel one another, leaving a smooth external force even while the velocity becomes unbounded.

To understand why this settles the Millennium Prize Problem, it is important to look at its precise formulation. The problem was not stated only as

> Do all smooth solutions of the unforced Navier–Stokes equations remain smooth forever?

Instead, the official formulation by Fefferman (Fefferman 2000) allows a solution in either direction. It lists four possible statements.

Statements (A) and (B) assert global smoothness for arbitrary smooth initial data with no external force, respectively on $\mathbb{R}^3$ and in the periodic setting.

Statements (C) and (D) ask for examples showing breakdown, respectively on $\mathbb{R}^3$ and in the periodic setting, and these formulations allow a smooth external force.

Thus proving even one of these four statements resolves the stated Millennium Prize Problem. The announced construction establishes statement (C), and OpenAI reports that the work also establishes the periodic alternative (D).

This leaves an interesting distinction. The classical unforced question

$$f \equiv 0$$

remains open: we still do not know whether arbitrary smooth, divergence-free initial velocity in three dimensions can develop a singularity without an external force. But the official Millennium Prize Problem was deliberately formulated more broadly, so a valid proof of (C) or (D) is sufficient.

The existence of weak solutions does not contradict the theorem. Leray proved already in 1934 that suitable weak solutions of the three-dimensional Navier–Stokes equations exist globally. A weak solution is allowed to have much less regularity than a classical smooth solution. The difficult question has always been whether such solutions must remain smooth. A finite-time blowup shows that, for the data in the new construction, the answer is no.

An unusual feature of the announcement is that the mathematical writeup was released together with a formal proof in the Lean proof assistant. A formal proof reduces the argument to a sequence of statements checked by a small trusted kernel, providing a form of verification very different from ordinary peer review.

Nevertheless, mathematical acceptance is a separate process. At the time of writing, the result has only just been released and is undergoing independent scrutiny. The Clay Mathematics Institute still lists the Navier–Stokes equation among the unsolved Millennium Prize Problems. Thus it is appropriate, for now, to describe this as an *announced solution* rather than an already established theorem of the mathematical literature.

If the proof withstands verification, the result will resolve one of the most famous problems in partial differential equations and provide the first rigorous example of finite-time singularity formation for the three-dimensional Navier–Stokes equations in the setting allowed by the Millennium Prize formulation.

## References

Fefferman, Charles L. 2000. “Existence and Smoothness of the Navier–Stokes Equation.” *The Millennium Prize Problems*.

OpenAI. 2026. *On the Navier–Stokes Millennium Prize Problem*. https://openai.com/index/navier-stokes-solution/.