Tutte's three-edge-colouring conjecture

Bogdan Grechuk • 7 Sep 2026

https://functor.network/user/3333/entry/1968

In August 2026, Yuta Inoue, Ken-ichi Kawarabayashi, Ritarou Matsuo, Atsuyuki Miyashita, Bojan Mohar, and Tomohiro Sonobe (Inoue et al. 2026) completed the proof of a famous conjecture of Tutte from 1966. The result is closely connected with the Four Colour Theorem and forms an important part of the still largely mysterious theory of nowhere-zero flows.

Let $G = (V, E)$ be a graph. An *orientation* of G is obtained by choosing a direction for every edge. For a vertex v, let $E^+(v)$ and $E^-(v)$ denote the sets of edges directed out of and into v, respectively.

For an integer $k > 1$, a *nowhere-zero k-flow* is an orientation of G together with a function

$$f : E \to \{1, \dots, k-1\}$$

such that

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$$

for every vertex v. This concept was introduced by Tutte (Tutte 1949).

A connected graph is *h-edge-connected* if it remains connected after deleting fewer than h edges. A graph is *bridgeless* if it has no edge whose deletion disconnects a component.

The *Petersen graph* is the graph on vertices

$$u_0, \dots, u_4, v_0, \dots, v_4,$$

with subscripts modulo 5, and edges

$$u_i u_{i+1}, \qquad u_i v_i, \qquad v_i v_{i+2}$$

for $i = 0, \ldots, 4$. Thus the u_i form a pentagon, the v_i form a five-pointed star, and each u_i is joined to v_i.

A graph H is a *minor* of G if H can be obtained from G by deleting vertices or edges and by contracting edges.

Tutte formulated three famous conjectures about nowhere-zero flows:

- *The 3-flow conjecture.* Every 4-edge-connected graph has a nowhere-zero 3-flow.
- *The 4-flow conjecture.* Every bridgeless graph without the Petersen graph as a minor has a nowhere-zero 4-flow.
- *The 5-flow conjecture.* Every bridgeless graph has a nowhere-zero 5-flow.

There is a pleasing balance between these three conjectures. The 3-flow conjecture asks for the smallest genuinely interesting flow value, but compensates by assuming strong connectivity. The 5-flow conjecture applies to the largest possible class of graphs: a graph containing a bridge cannot have a nowhere-zero flow of any size. The number 5 cannot simply be replaced by 4, because the Petersen graph is bridgeless but has no nowhere-zero 4-flow.

The 4-flow conjecture sits naturally between these two extremes. It predicts that the Petersen graph is, in a precise minor-theoretic sense, the only essential obstruction to a nowhere-zero 4-flow.

All three conjectures remain open. There are, however, very strong partial results. Jaeger proved that every 4-edge-connected graph has a nowhere-zero 4-flow (Jaeger 1979). Thomassen proved that every 8-edge-connected graph has a nowhere-zero 3-flow (Thomassen 2012), and Lov’asz, Thomassen, Wu, and Zhang improved 8 to 6 (Lovász et al. 2013):

$$\text{Every 6-edge-connected graph has a nowhere-zero 3-flow.}$$

Kochol showed that proving this for 5-edge-connected graphs would already imply the full 3-flow conjecture (Kochol 2001).

For the 5-flow conjecture, Seymour proved in 1981 (Seymour 1981) that

$$\text{Every bridgeless graph has a nowhere-zero 6-flow.}$$

The 2026 result concerns the 4-flow conjecture for *cubic graphs*, that is, graphs in which every vertex has degree 3.

A *proper* 3*-edge-colouring* assigns one of three colours to every edge so that edges meeting at a common vertex receive different colours. For cubic graphs,

$$G \text{ has a nowhere-zero 4-flow} \iff G \text{ is 3-edge-colourable.}$$

Indeed, one may identify the three colours with the three nonzero elements of

$$\mathbb{Z}_2 \times \mathbb{Z}_2.$$

Thus the cubic case of Tutte's 4-flow conjecture becomes the following statement.

Tutte's three-edge-colouring conjecture. Every 2-connected cubic graph without the Petersen graph as a minor is 3-edge-colourable.

Here a connected graph with at least three vertices is 2*-connected* if deleting any one vertex leaves it connected.

The conjecture is closely related to the Four Colour Theorem. One equivalent form of that theorem says that every 2-connected cubic planar graph is 3-edge-colourable. Tutte's conjecture replaces planarity by the much weaker condition of excluding the Petersen graph as a minor.

In 1997, Robertson, Seymour, and Thomas (Robertson et al. 1997) reduced the conjecture to two special classes of graphs.

A graph is *apex* if deleting one vertex makes it planar. A graph is *doublecross* if it can be drawn in the plane with only two crossings, both on the boundary of the same face.

They showed that it is enough to prove:

- every 2-connected doublecross cubic graph is 3-edge-colourable;
- every 2-connected apex cubic graph is 3-edge-colourable.

The doublecross case was proved by Edwards, Sanders, Seymour, and Thomas (Edwards et al. 2016). The apex case remained.

In 2026, Inoue, Kawarabayashi, Matsuo, Miyashita, Mohar, and Sonobe proved (Inoue et al. 2026):

Theorem. Every 2-connected apex cubic graph is 3-edge-colourable.

Together with the previous reduction and the doublecross case, this proves Tutte's three-edge-colouring conjecture.

Equivalently,

$$\boxed{\text{Every bridgeless cubic graph without a Petersen minor has a nowhere-zero}}$$

The proof is computer-assisted and uses the same broad ideas of reducible configurations and discharging that appear in the Four Colour Theorem. It is also constructive: the authors obtain an $O(n^2)$ algorithm for finding a 3-edge-colouring of an apex cubic graph.

It is important that this does *not* prove Tutte's full 4-flow conjecture. The general 3-flow, 4-flow, and 5-flow conjectures all remain open. What has now been settled is the cubic case of the 4-flow conjecture, sixty years after Tutte formulated it.

References

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