Long gaps between primes

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https://functor.network/user/3333/entry/1964

Yesterday, I wrote about small gaps between primes. Today, let me write about long gaps.

If we denote by

$$G(X) = \sup_{p_{n+1} \leq X} (p_{n+1} - p_n) \tag{1}$$

the largest gap between consecutive primes up to X, then the prime number theorem implies that

$$G(X) \geq (1 + o(1)) \log X.$$

All known methods for improving this lower bound proceed by constructing a sequence of consecutive integers, each of which has a "very small" prime factor. More formally, let $j(n)$ be the maximal gap between consecutive integers coprime to n, and let

$$J(X) = \max_{n \leq X} j(n).$$

Then it is easy to see that $G(2X) \geq J(X)$ for all $X \geq 3$. Hence, any lower bound on $J(X)$ implies a corresponding lower bound for $G(X)$.

Maier and Pomerance (Maier and Pomerance 1990) conjectured in 1990 that

$$J(X) = \log X (\log \log X)^{2+o(1)},$$

and this conjecture is widely believed. If it is true, then a lower bound of the form

$$G(X) \geq \log X (\log \log X)^{2+o(1)}$$

is the conjectural limit of the current method. Nevertheless, it took more than 90 years, and eventually the help of AI, to reach this bound.

In 1931, Westzynthius (Westzynthius 1931) was the first to prove that

$$\lim_{X \to \infty} \frac{G(X)}{\log X} = \infty.$$

In 1935, Erdős (Erdős 1935) proved the lower bound

$$G(X) \geq \log X (\log\log X)^{1+o(1)}.$$

At this level of precision, the exponent $1 + o(1)$ stood for more than 90 years, despite important improvements to the lower-order terms by many authors, including Rankin, Ford, Green, Konyagin, Tao, and Maynard (Rankin 1938; Ford et al. 2016, 2018; Maynard 2016).

On September 3, 2026, OpenAI published a proof, due to GPT-6 Astra, of the improved bound

$$G(X) \geq \log X (\log\log X)^{2+o(1)}.$$

As mentioned above, up to the $o(1)$ term, this reaches the conjectural limit of the current method.

Cramér's random model of the primes predicts maximal gaps on the scale $(\log X)^2$. In fact, the model itself gives the constant 1, and Shanks later conjectured the asymptotic

$$G(X) \sim (\log X)^2.$$

More refined modern probabilistic models (Banks et al. 2023) suggest instead the constant

$$2e^{-\gamma} = 1.1229\ldots,$$

subject to an additional natural conjecture about the underlying sieve problem.

The best published upper bound remains

$$G(X) < X^{0.525}$$

for all sufficiently large X, proved by Baker, Harman and Pintz (Baker et al. 2001) in 2001.

References

Baker, Roger C., Glyn Harman, and János Pintz. 2001. "The Difference Between Consecutive Primes, II." *Proc. Lond. Math. Soc.* 83 (3): 532–62.

Banks, William, Kevin Ford, and Terence Tao. 2023. "Large Prime Gaps and Probabilistic Models." *Invent. Math.* 233 (3): 1471–518. https://doi.org/10.1007/s00222-023-01199-0.

Erdős, Paul. 1935. "On the Difference of Consecutive Primes." *Q. J. Math.* 6 (1): 124–28.

Ford, Kevin, Ben Green, Sergei Konyagin, James Maynard, and Terence Tao. 2018. "Long Gaps Between Primes." *J. Amer. Math. Soc.* 31 (1): 65–105.

Ford, Kevin, Ben Green, Sergei Konyagin, and Terence Tao. 2016. “Large Gaps Between Consecutive Prime Numbers.” *Ann. of Math.*, 935–74.
Maier, Helmut, and Carl Pomerance. 1990. “Unusually Large Gaps Between Consecutive Primes.” *Trans. Amer. Math. Soc.* 322 (1): 201–37.
Maynard, James. 2016. “Large Gaps Between Primes.” *Ann. of Math.* 183 (3): 915–33.
Rankin, R. A. 1938. “The Difference Between Consecutive Prime Numbers.” *J. Lond. Math. Soc.* s1-13 (4): 242–47. https://doi.org/10.1112/jlms/s1-13.4.242.
Westzynthius, Erik Johan. 1931. “Über die Verteilung der Zahlen: die zu den n ersten Primzahlen teilerfremd sind.” PhD thesis, Helsingfors.