Small gaps between primes

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https://functor.network/user/3333/entry/1963

This post continues my series on favorite theorems of the twenty-first century. For an overview of the categories and my earlier selections, see this introductory post.

My choice for 2005 in number theory is the theorem of Goldston, Pintz, and Yıldırım on small gaps between consecutive primes. The result was proved and submitted in 2005, although the final paper appeared in the *Annals of Mathematics* in 2009 (Goldston et al. 2009).

Let

$$2 = p_1 < p_2 < p_3 < \cdots$$

be the sequence of prime numbers. One of the oldest and most famous unsolved problems in number theory is the twin prime conjecture, which predicts that

$$p_{n+1} - p_n = 2$$

for infinitely many n.

The prime number theorem implies that the average distance between primes around a large number x is about $\log x$. Thus, around p_n, the natural scale for the gap $p_{n+1} - p_n$ is $\log p_n$. This suggests studying

$$\Delta = \liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n}.$$

The prime number theorem immediately gives $\Delta \leq 1$. In 1940, Erdős made the first unconditional improvement, proving that $\Delta < 1$ (Erdős 1940). During the following decades many authors pushed the upper bound downward. By 2005, the record was Maier's

$$\Delta \leq 0.2484\ldots$$

(Maier 1988).

It was natural to conjecture that the correct answer is actually zero. Goldston, Pintz, and Yıldırım finally proved this.

Theorem 1 ***(Goldston–Pintz–Yıldırım).***

$$\boxed{\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0.}$$

Equivalently, for every $\epsilon > 0$ there are infinitely many n such that

$$p_{n+1} - p_n < \epsilon \log p_n.$$

Thus consecutive primes can be closer than *any prescribed fraction* of their average spacing.

The proof introduced many novel ideas which lead to further progress. The same authors subsequently pushed their method much further. In (Goldston et al. 2010) they proved

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\sqrt{\log p_n}(\log\log p_n)^2} < \infty.$$

Thus infinitely many prime gaps are not merely an arbitrarily small fraction of the average gap: they are almost as small as the square root of the average gap.

For several years bounded gaps nevertheless remained just out of reach. Then, in 2013, Yitang Zhang found exactly the sort of additional distributional information that the GPY framework could exploit. He proved

$$\liminf_{n\to\infty} (p_{n+1} - p_n) < 70\,000\,000,$$

the first unconditional proof that some fixed finite bound works infinitely often (Zhang 2014). This was a spectacular breakthrough.

Progress then became extremely rapid. James Maynard, independently of related work of Terence Tao, developed a more flexible multidimensional version of the sieve and obtained the bound

$$\liminf_{n\to\infty} (p_{n+1} - p_n) \le 600$$

(Maynard 2015). The Polymath8 project combined and optimized the new ideas and eventually reduced the unconditional bound to

$$\liminf_{n\to\infty}(p_{n+1}-p_n)\le 246$$

(D. H. J. Polymath 2014).

Very recently this story has taken another remarkable turn. The OpenAI project *PrimeGaps186* reports the further bound

$$\boxed{\liminf_{n\to\infty}(p_{n+1}-p_n)\le 186}$$

(OpenAI 2026). In other words, there exist infinitely many pairs of consecutive primes are separated by at most 186. Its accompanying repository contains a Lean 4 formalization.

The twin prime conjecture would replace 186 by 2, so a substantial gap remains. Yet the route from the state of knowledge before 2005 to the present is extraordinary. This is one reason Theorem 1 is my favorite result of that year in number theory. It did not prove the twin prime conjecture, nor even bounded gaps. Instead, it discovered the mechanism that transformed the problem and lead to remarkable further developments.

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