Supercritical sharpness of percolation

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In March 2026, Sahar Diskin, Philip Easo, Ritvik Ramanan Radhakrishnan, Benny Sudakov and Vincent Tassion posted a preprint (Diskin et al. 2026) proving a general theorem about supercritical percolation on infinite transitive graphs. The theorem determines, up to constant factors in the exponent, the probability that a given vertex belongs to a large but finite connected component. Remarkably, the answer is governed by a purely geometric quantity: the isoperimetric function of the graph.

Let $G = (V, E)$ be an infinite connected locally finite graph. Here "locally finite" means that every vertex has finite degree. We say that G is *transitive* if, for every two vertices $u, v \in V$, there is an automorphism of G sending u to v. Informally, a transitive graph looks the same from every vertex. The standard lattice $\mathbb{Z}^d$ is a basic example.

In *Bernoulli bond percolation* with parameter $p \in [0, 1]$, every edge of G is declared open independently with probability p, and closed otherwise. The open edges form a random subgraph of G. For a vertex v, let C_v denote the set of vertices in the open connected component containing v.

As p increases, a phase transition occurs. Define

$$p_c(G) := \inf\{p \in [0, 1] : \mathbb{P}_p(|C_v| = \infty) > 0\}.$$

By transitivity, this number does not depend on the choice of v. If $p < p_c(G)$, all open components are finite almost surely. If $p > p_c(G)$, an infinite open component exists with positive probability. This is called the *supercritical regime*.

Even in the supercritical regime, the component C_v can still be finite. How unlikely is it that it is finite but very large? More precisely, how fast does

$$\mathbb{P}_p(n \le |C_v| < \infty)$$

decay as n tends to infinity?

The answer depends on how efficiently a finite set of vertices can be separated from the rest of the graph. For a finite set $S \subset V$, let

$$\partial_E S := \{e \in E : \ e \text{ has exactly one endpoint in } S\}$$

be its *edge boundary*. Define the *isoperimetric function* by

$$\Phi(n) := \min\{|\partial_E S| : S \subset V, \ n \le |S| < \infty\}.$$

Thus, $\Phi(n)$ is the smallest possible boundary of a finite set containing at least n vertices.

Theorem (Diskin–Easo–Radhakrishnan–Sudakov–Tassion). *Let G be an infinite connected locally finite transitive graph. For every $p > p_c(G)$ there exists a constant $c > 0$ such that, for every vertex $v \in V$ and every $n \ge 1$,*

$$\mathbb{P}_p(n \le |C_v| < \infty) \le \exp(-c\Phi(n)).$$

The estimate is sharp up to the value of the constant in the exponent. For every $p > p_c(G)$ there exists $C > 0$ such that

$$\mathbb{P}_p(n \le |C_v| < \infty) \ge \exp(-C\Phi(n)).$$

Thus the isoperimetric function gives exactly the right scale for the logarithm of the probability of a large finite component.

The lattice $\mathbb{Z}^d$ gives a particularly transparent example. Its isoperimetric function has order

$$\Phi(n) \approx n^{(d-1)/d}.$$

Consequently, for every $p > p_c(\mathbb{Z}^d)$, there exist positive constants c and C such that

$$\exp(-Cn^{(d-1)/d}) \le \mathbb{P}_p(n \le |C_v| < \infty) \le \exp(-cn^{(d-1)/d}).$$

For instance, in two dimensions the exponent has order $\sqrt{n}$, while in three dimensions it has order $n^{2/3}$.

There is a simple geometric reason why a boundary rather than a volume should appear. In the supercritical regime, open edges are sufficiently abundant to create an infinite component. If C_v is nevertheless finite, it must somehow be cut off from infinity. A component containing about n vertices can be surrounded using roughly $\Phi(n)$ boundary edges, and requiring the relevant connections across this boundary to fail naturally has probability exponential in the size of the boundary. The theorem says that this heuristic gives the correct answer on every infinite transitive graph.

The proof makes this intuition precise using a technique known as *sprinkling*. Choose

$$p_c(G) < q < p.$$

One may first generate a percolation with parameter q and then independently open some additional edges so that the resulting configuration has parameter p. This second random step is called sprinkling.

The main technical achievement of (Diskin et al. 2026) is to show that if a finite set S has large isoperimetric boundary, then the q-percolation provides many opportunities for S to connect to infinity. The authors call such opportunities “touches”. They construct these touches one after another by an exploration procedure. Roughly speaking, as long as fewer than a constant multiple of $\Phi(S)$ touches have been found, another one can be produced with probability close to 1.

Now suppose that $S = C_v$ is finite in the final percolation with parameter p. All these potential connections to infinity must fail during sprinkling; otherwise C_v would become part of an infinite component. Since the sprinkling is independent, the probability that all these opportunities fail decreases exponentially in their number. Since the number of opportunities is proportional to $\Phi(S)$, this leads to the factor

$$\exp(-c\Phi(n))$$

in the theorem.

It is useful to compare the result with what was previously known. A transitive graph is called *nonamenable* if there exists $a > 0$ such that

$$|\partial_E S| \geq a|S|$$

for every finite set $S \subset V$. In this case

$$\Phi(n) \geq an,$$

so the new theorem gives

$$\mathbb{P}_p(n \leq |C_v| < \infty) \leq \exp(-cn).$$

Such an exponential estimate was previously proved by Hermon and Hutchcroft (Hermon and Hutchcroft 2021) for nonamenable quasi-transitive graphs.

At the opposite end are amenable graphs, for which finite sets may have boundary much smaller than their volume. The lattices $\mathbb{Z}^d$ are the most familiar examples. Supercritical sharpness had previously been established for transitive graphs of polynomial growth and for nonamenable transitive graphs. The theorem of Diskin, Easo, Ramanan Radhakrishnan, Sudakov and Tassion removes these restrictions and applies to every infinite transitive graph.

There is still one conspicuous point not covered by the result: the critical value $p = p_c(G)$. Understanding percolation exactly at criticality is usually much more difficult, and many fundamental questions remain open there.

References

Diskin, Sahar, Philip Easo, Ritvik Ramanan Radhakrishnan, Benny Sudakov, and Vincent Tassion. 2026. “Supercritical Sharpness of Percolation.” *arXiv Preprint arXiv:2603.03257*.

Hermon, Jonathan, and Tom Hutchcroft. 2021. “Supercritical Percolation on Nonamenable Graphs: Isoperimetry, Analyticity, and Exponential Decay of the Cluster Size Distribution.” *Invent. Math.* 224 (2): 445–86.