The Schinzel–Zassenhaus conjecture is proved

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An important paper by Vesselin Dimitrov (Dimitrov 2019), first posted online in 2019, has now been accepted for publication in the *Annals of Mathematics*. The paper proves the Schinzel–Zassenhaus conjecture, a long-standing and remarkably simple-to-state conjecture about the location of the roots of integer polynomials.

Let $\mathbb{Z}[x]$ denote the ring of polynomials in one variable with integer coefficients. A polynomial $P \in \mathbb{Z}[x]$ is *irreducible* if it cannot be written as a product

$$P(x) = Q(x)R(x)$$

with both $Q, R \in \mathbb{Z}[x]$ non-constant. An irreducible polynomial $P \in \mathbb{Z}[x]$ is called *cyclotomic* if it divides $x^m - 1$ for some integer $m \geq 1$. Thus all roots of a cyclotomic polynomial are roots of unity, and in particular they all lie on the unit circle.

The Schinzel–Zassenhaus conjecture concerns what happens for irreducible polynomials that are not cyclotomic. Let P be a monic, irreducible, non-cyclotomic polynomial of degree n. Since the roots of a cyclotomic polynomial all have absolute value 1, one expects a genuinely non-cyclotomic polynomial to have a root that moves away from the unit circle. The conjecture asserts that this movement cannot be arbitrarily small compared with the degree: there should exist an absolute constant $c > 0$ such that P has a root α satisfying

$$|\alpha| \geq 1 + \frac{c}{n}.$$

The question goes back to the work of A. Schinzel and H. Zassenhaus in 1965 (Schinzel and Zassenhaus 1965). Their results gave an explicit, but much weaker, bound. They also observed that the natural scale of the problem should be of order $1/n$. In other words, the central issue was to prove a lower bound that remains uniformly of size $1/n$ as the degree grows.

There has been substantial progress on this problem in special families. For example, in 2007, Peter Borwein, Edward Dobrowolski, and Michael J. Mossinghoff proved the conjecture for polynomials with odd coefficients, and more generally for certain congruence classes of coefficients (Borwein et al. 2007). The general case, however, remained open.

In 2019, Dimitrov posted a preprint giving a proof in full generality (Dimitrov 2019). The result is completely explicit:

Theorem 1 *Every non-cyclotomic monic irreducible polynomial $P(x) \in \mathbb{Z}[x]$ of degree $n > 1$ has a root $\alpha \in \mathbb{C}$ satisfying*

$$|\alpha| \geq 2^{1/(4n)}.$$

This is stronger than merely asserting the existence of some universal constant c. Indeed,

$$2^{1/(4n)} = \exp\left(\frac{\log 2}{4n}\right) \geq 1 + \frac{\log 2}{4n},$$

and hence one may take

$$c = \frac{\log 2}{4}.$$

Thus the conjectured $1/n$-scale separation from the unit circle holds uniformly for every degree.

The theorem can also be viewed as a particularly clean dichotomy: cyclotomic polynomials have all their roots exactly on the unit circle, while every other monic irreducible integer polynomial of degree $n > 1$ must have a root outside the circle of radius 1 by the explicit amount encoded in $2^{1/(4n)}$.

The acceptance of Dimitrov's paper by the *Annals of Mathematics* marks the culmination of a problem that originated more than sixty years ago and illustrates how a very elementary-looking question about polynomial roots can lead to deep arithmetic ideas.

References

Borwein, Peter, Edward Dobrowolski, and Michael J. Mossinghoff. 2007. "Lehmer's Problem for Polynomials with Odd Coefficients." *Annals of Mathematics* 166 (2): 347–66. https://doi.org/10.4007/annals.2007.166.347.

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