Emanuel Milman’s talk on multi-bubble isoperimetric problems

Bogdan Grechuk • 27 Aug 2026

This post continues our series of online seminars devoted to accessible presentations of some of the most significant mathematical theorems of the 21st century.

I am delighted to report that on 19th August 2026, Prof. Marc Lackenby delivered a clear and accessible lecture on knots, links, and polynomial bounds. If you missed the talk, the recording is now available on YouTube.

The next lecture in the series will take place on

Wednesday, 16th September 2026, at 4:00 PM UK time.

Prof. Emanuel Milman (Technion–Israel Institute of Technology, Haifa, Israel) will speak about multi-bubble isoperimetric problems. The lecture will describe the history of these problems and several major recent advances in Euclidean and spherical spaces, concerning double, triple, quadruple, and quintuple bubbles. It will also explain some of the geometric ideas behind the proofs and discuss the main questions that remain open.

To join the talk, please follow the meeting link on the (or use this direct link) at the scheduled time; no registration is required. The same webpage contains the full list of upcoming seminars.

For announcements of other talks, as well as descriptions of recent major mathematical breakthroughs, see the full list of posts on this blog.

Here is some background on the mathematics of the talk.

The classical isoperimetric theorem says that among all regions in $\mathbb{R}^d$ of a given volume, the ball has the smallest possible boundary area. A natural generalization asks what happens when several regions of prescribed volumes must be enclosed and separated simultaneously.

More precisely, suppose that positive numbers

$$V_1, \ldots, V_k$$

are given. We seek pairwise disjoint regions in $\mathbb{R}^d$ having these volumes and minimizing the total area of all interfaces, with an interface shared by two regions counted only once. It may be advantageous for the regions to touch, since they can then share part of their boundary.

The first non-trivial case is $k = 2$. A *standard double bubble* consists of three pieces of spheres, one of which may be flat, meeting at angles of $120°$. In $\mathbb{R}^3$ the three pieces meet along a common circle. When the two prescribed volumes are equal, the surface separating them is a flat disc.

The planar double-bubble problem was solved in 1993 by Joel Foisy, Manuel Alfaro Garcia, Jeffrey Brock, Nickelous Hodges, and Jason Zimba, all of whom were undergraduate students at the time (Foisy et al. 1993). In 2002, Michael Hutchings, Frank Morgan, Manuel Ritoré, and Antonio Ros proved the three-dimensional result (Hutchings et al. 2002). Ben Reichardt subsequently extended it to every dimension (Reichardt 2008).

Theorem 1 *For every $d \geq 2$ and every pair of positive volumes V_1, V_2, the standard double bubble minimizes the total boundary area among all configurations enclosing and separating regions of volumes V_1 and V_2 in $\mathbb{R}^d$.*

There is a natural candidate for the optimal configuration with more bubbles. Choose $k + 1$ equidistant points on the unit sphere $\mathbb{S}^d$ and divide the sphere into their Voronoi cells: each point of $\mathbb{S}^d$ is assigned to the nearest of the chosen points. After a suitable Möbius transformation and stereographic projection to $\mathbb{R}^d$, one cell becomes the unbounded exterior region, while the remaining k cells form what is called a *standard k-bubble*.

For every $k \leq d + 1$ and every list of positive volumes, the parameters in this construction can be chosen so that the bounded regions have precisely the prescribed volumes (Montesinos Amilibia 2001). In 1996, John Sullivan conjectured that these standard bubbles always minimize the total boundary area (Sullivan and Morgan 1996). Thus, the isoperimetric theorem is the case $k = 1$, while the double-bubble theorem settles the case $k = 2$.

The planar triple-bubble problem was resolved by Wacharin Wichiramala in 2004 (Wichiramala 2004). For many years, however, even the triple-bubble problem in $\mathbb{R}^3$ remained open.

An important breakthrough came from a closely related problem involving Gaussian rather than ordinary volume. Recall that the standard Gaussian probability measure on $\mathbb{R}^d$ is

$$d\gamma_d(x) = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{|x|^2}{2}\right) dx.$$

Thus, regions near the origin receive more weight than equally large regions far away from it. Gaussian perimeter is defined similarly by weighting boundary area with the Gaussian density.

Suppose that $\mathbb{R}^d$ is to be partitioned into q regions having prescribed Gaussian measures

$$v_1, \ldots, v_q > 0, \qquad v_1 + \cdots + v_q = 1.$$

A natural candidate is a simplicial partition obtained from the Voronoi cells of q equidistant points, suitably positioned relative to the origin.

In 2022, Emanuel Milman and Joe Neeman proved that this candidate is indeed optimal throughout the full possible range (Milman and Neeman 2022).

Theorem 2 *Let $d \geq 2$ and $2 \leq q \leq d+1$. Among all partitions of $\mathbb{R}^d$ into q regions of prescribed Gaussian measures, the least-Gaussian-perimeter partition is given by the Voronoi cells of q equidistant points, suitably translated. Moreover, the minimizing partition is unique up to sets of Gaussian measure zero.*

The proof required ideas quite different from those used for the classical double-bubble theorem. In particular, Milman and Neeman developed a matrix-valued differential inequality for the Gaussian isoperimetric profile and combined it with a detailed analysis of stable minimizing clusters.

These methods soon led back to the original Euclidean problem. In work published in 2025, Milman and Neeman established the triple- and quadruple-bubble cases of Sullivan's conjecture in the relevant dimensions, together with their spherical analogues (Milman and Neeman 2025).

Theorem 3 *For every $d \geq 3$, the standard triple bubble minimizes the total boundary area among all configurations enclosing three prescribed volumes in $\mathbb{R}^d$. For every $d \geq 4$, the analogous statement holds for four prescribed volumes. In both cases the minimizers are unique, up to the natural symmetries and sets of measure zero.*

Together with Wichiramala's planar result, this settles the triple-bubble problem in every dimension $d \geq 2$. For four bubbles, the main unresolved Euclidean dimensions are $d = 2$ and $d = 3$. The planar problem falls outside the range of Sullivan's conjecture, although the case of four equal areas was solved by Emanuele Paolini and Vincenzo Tortorelli (Paolini and Tortorelli 2020). The standard quadruple bubble in $\mathbb{R}^3$ remains one of the most prominent open cases.

Milman and Neeman have also proved the quintuple-bubble theorem in dimensions $d \geq 5$ (Milman and Neeman 2023). On the sphere they obtained both optimality and uniqueness. In Euclidean space they proved that the standard quintuple bubble is a minimizer, although uniqueness is not yet known.

A further recent result of Milman and Botong Xu concerns *stability* (Milman and Xu 2025). They proved that standard k-bubbles in the principal constant-curvature spaces are stable whenever $2 \leq k \leq d+1$. Here stability means that every smooth, volume-preserving infinitesimal deformation has non-negative second variation of perimeter. This is a necessary local optimality property, although by itself it does not prove global minimality.

The story therefore begins with the familiar fact that a single ball is optimal, passes through the double-bubble theorem, and leads to recent results about triple, quadruple, and quintuple bubbles. The remaining problems include some configurations that are very easy to draw but have resisted all known methods. The lecture will explain both this remarkable sequence of advances and the geometric obstacles that still stand in the way of a complete solution.

References

Foisy, Joel, Manuel Alfaro Garcia, Jeffrey Farlowe Brock, Nickelous Hodges, and Jason Zimba. 1993. “The Standard Double Soap Bubble in $\mathbb{R}^2$ Uniquely Minimizes Perimeter.” *Pacific Journal of Mathematics* 159 (1): 47–59. https://doi.org/10.2140/pjm.1993.159.47.

Hutchings, Michael, Frank Morgan, Manuel Ritoré, and Antonio Ros. 2002. “Proof of the Double Bubble Conjecture.” *Annals of Mathematics* 155 (2): 459–89. https://doi.org/10.2307/3062123.

Milman, Emanuel, and Joe Neeman. 2022. “The Gaussian Double-Bubble and Multi-Bubble Conjectures.” *Annals of Mathematics* 195 (1): 89–206. https://doi.org/10.4007/annals.2022.195.1.2.

Milman, Emanuel, and Joe Neeman. 2023. *Plateau Bubbles and the Quintuple Bubble Theorem on* $\mathbb{S}^n$. https://doi.org/10.48550/arXiv.2307.08164.

Milman, Emanuel, and Joe Neeman. 2025. “The Structure of Isoperimetric Bubbles on $\mathbb{R}^n$ and $\mathbb{S}^n$.” *Acta Mathematica* 234 (1): 71–188. https://doi.org/10.4310/ACTA.2025.v234.n1.a2.

Milman, Emanuel, and Botong Xu. 2025. *Standard Bubbles (and Other Möbius-Flat Partitions) on Model Spaces Are Stable*. https://doi.org/10.48550/arXiv.2504.11185.

Montesinos Amilibia, A. 2001. “Existence and Uniqueness of Standard Bubble Clusters of Given Volumes in $\mathbb{R}^N$.” *Asian Journal of Mathematics* 5 (1): 25–31. https://doi.org/10.4310/AJM.2001.v5.n1.a3.

Paolini, Emanuele, and Vincenzo M. Tortorelli. 2020. “The Quadruple Planar Bubble Enclosing Equal Areas Is Symmetric.” *Calculus of Variations and Partial Differential Equations* 59 (1). https://doi.org/10.1007/s00526-019-1687-9.

Reichardt, Ben W. 2008. “Proof of the Double Bubble Conjecture in $\mathbb{R}^n$.” *The Journal of Geometric Analysis* 18 (1): 172–91. https://doi.org/10.1007/s12220-007-9002-y.

Sullivan, John M., and Frank Morgan. 1996. “Open Problems in Soap Bubble Geometry.” *International Journal of Mathematics* 7 (6): 833–42. https://doi.org/10.1142/S0129167X9600044X.

Wichiramala, Wacharin. 2004. “Proof of the Planar Triple Bubble Conjecture.” *Journal für Die Reine Und Angewandte Mathematik* 567: 1–49. https://doi.org/10.1515/crll.2004.011.