The Lawson–Osserman conjecture is true in 2D

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A paper (Hirsch et al. 2026) by Jonas Hirsch, Connor Mooney and Riccardo Tione has recently been accepted for publication in *Inventiones Mathematicae*. The authors solve the planar case of the Lawson–Osserman conjecture and, in fact, prove a stronger regularity theorem: every Lipschitz weak solution of the minimal surface system on a two-dimensional domain is smooth.

Let

$$B_1 = \{x \in \mathbb{R}^2 \mid |x| < 1\}$$

be the open unit disk, and let

$$u = (u^1, \dots, u^n) : B_1 \to \mathbb{R}^n$$

be a Lipschitz function. This means that there is a constant L such that

$$|u(x) - u(y)| \le L|x - y| \qquad \text{for all } x, y \in B_1.$$

The graph of u is the two-dimensional surface

$$\Gamma_u = \{(x, u(x)) \in \mathbb{R}^{n+2} \mid x \in B_1\}.$$

The number n is the codimension of this surface.

A Lipschitz function is differentiable almost everywhere. At points where u is differentiable, let Du be its $n \times 2$ derivative matrix. The area of the graph is

$$\mathcal{E}(u) = \int_{B_1} \sqrt{\det\left(I_2 + (Du)^T Du\right)}\, \mathrm{d}x,$$

where I_2 is the 2×2 identity matrix.

We say that u is *outer critical* for the area functional if

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{E}(u + t\varphi) = 0$$

for every smooth function

$$\varphi : B_1 \to \mathbb{R}^n$$

which vanishes near the boundary of B_1. In other words, no sufficiently small deformation of the values of u changes the area to first order. An outer-critical Lipschitz function is also called a *Lipschitz weak solution of the minimal surface system*.

When $n = 1$, the minimal surface system is the familiar single equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

For $n > 1$, however, one obtains a coupled nonlinear system of equations. Such systems can have much worse regularity properties than single equations.

Hirsch, Mooney and Tione proved the following.

Theorem 1 *Let $n \geq 1$, and let $u : B_1 \to \mathbb{R}^n$ be Lipschitz. If u is outer critical for the area functional, then*

$$u \in C^\infty(B_1, \mathbb{R}^n).$$

Thus, although u is initially assumed to have only bounded first derivatives almost everywhere, the minimal surface system forces all derivatives of every order to exist and be continuous. A Lipschitz minimal graph over a two-dimensional domain cannot hide corners, creases or other interior singularities.

It is important that the theorem does not assume that Γ_u minimizes area among all competing surfaces. The much weaker condition of being critical to first order is sufficient. The conclusion is an interior statement: it does not say that u must remain smooth up to the boundary of B_1.

Let us now explain the connection with the Lawson–Osserman conjecture. Besides changing the values of u, one can deform the points of its domain. For a smooth vector field

$$\Phi : B_1 \to \mathbb{R}^2$$

which vanishes near the boundary, consider

$$u \circ (\mathrm{id}_{B_1} + t\Phi).$$

A map is called *inner critical* if such deformations also leave its area unchanged to first order, and it is called *stationary* if it is both outer and inner critical.

For twice continuously differentiable maps, outer criticality automatically implies inner criticality. Lawson and Osserman conjectured in 1977 that this remains true for Lipschitz maps. The new theorem proves much more in dimension two: an outer-critical Lipschitz map is automatically smooth, and therefore automatically stationary.

The result was classical in codimension one, when $n = 1$. The striking point is that the new theorem allows n to be arbitrary. The graph may lie in a Euclidean space of any dimension, but as long as its domain is two-dimensional, weak criticality forces complete smoothness.

The proof uses several special features of two-dimensional geometry. After a suitable change of coordinates, the authors write

$$u = v \circ \phi,$$

where v is harmonic and ϕ is a controlled deformation of the plane. Away from isolated exceptional points, the derivative of v has rank either one or two. Near rank-one points, the area functional behaves like a uniformly convex functional, and standard regularity theory applies. Near rank-two points, two components of u can be used as local coordinates, reducing the problem to regularity results related to the Monge–Ampère equation. The isolated exceptional points are then shown to be removable.

Every step uses the fact that the domain has dimension two. What happens for Lipschitz weak solutions of the minimal surface system in higher-dimensional domains remains largely mysterious.

This theorem is a beautiful example of hidden regularity: a map which appears at first to be merely Lipschitz is forced by a geometric system of weak equations to have been smooth all along.

References

Hirsch, Jonas, Connor Mooney, and Riccardo Tione. 2026. “On the Lawson–Osserman Conjecture.” *Inventiones Mathematicae*, 1–26.