The disconjugacy conjecture

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The paper by Steven Karp and Kevin Purbhoo (Karp and Purbhoo 2023), first posted online in 2023, has now been acceped for publication in the Journal of the American Mathematical Society. Among its applications is the resolution of the disconjugacy conjecture, formulated by Alexandre Eremenko in 2015. The statement involves only real polynomials, derivatives, determinants, and zeros.

Let $V \subset \mathbb{R}[x]$ be a d-dimensional vector space of polynomials, and let $f_1, \dots, f_d$ be any basis of V. The *Wronskian* of this basis is the polynomial

$$\mathrm{Wr}(f_1, \dots, f_d)(x) = \det \begin{pmatrix} f_1(x) & f_1'(x) & \cdots & f_1^{(d-1)}(x) \\ f_2(x) & f_2'(x) & \cdots & f_2^{(d-1)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_d(x) & f_d'(x) & \cdots & f_d^{(d-1)}(x) \end{pmatrix}.$$

Replacing $f_1, \dots, f_d$ by another basis multiplies this polynomial by a non-zero constant. Its zeros, together with their multiplicities, therefore depend only on V. We denote any such polynomial by $\mathrm{Wr}(V)$.

Recall that a real number a is a zero of multiplicity r of a non-zero polynomial f if

$$f(a) = f'(a) = \cdots = f^{(r-1)}(a) = 0, \qquad f^{(r)}(a) \neq 0.$$

The definition of the Wronskian implies that

$$\mathrm{Wr}(V)(a) = 0$$

if and only if V contains a non-zero polynomial having a zero at a of multiplicity at least d.

Let $I \subset \mathbb{R}$ be an interval. We say that V is *disconjugate on* I if every non-zero polynomial $f \in V$ has at most $d-1$ zeros in I, counted with multiplicity. The simplest example is the space of all polynomials of degree at most $d-1$, which is disconjugate on the whole real line.

If V is disconjugate on I, then $\mathrm{Wr}(V)$ clearly cannot vanish anywhere in I. It is tempting to guess that the converse is also true, but this fails without an additional assumption. For example, let

$$V_- = \{a(x^2-1) + bx : a, b \in \mathbb{R}\}.$$

Using the basis x^2-1, x, we obtain

$$\mathrm{Wr}(V_-)(x) = \det \begin{pmatrix} x^2-1 & 2x \\ x & 1 \end{pmatrix} = -(x^2+1).$$

Thus $\mathrm{Wr}(V_-)$ has no real zeros. Nevertheless, $x^2-1 \in V_-$ has two real zeros, so the two-dimensional space V_- is not disconjugate on $\mathbb{R}$.

The Wronskian in this example has two non-real zeros, namely $\pm i$. Eremenko conjectured that such counterexamples disappear if all complex zeros of the Wronskian are real (Eremenko 2015). The conjecture had previously been verified when $d \leq 2$, and Karp and Purbhoo proved it in full.

Theorem 1 (Karp–Purbhoo) *Let $V \subset \mathbb{R}[x]$ be a finite-dimensional vector space of polynomials, and let $d = \dim V$. Suppose that every complex zero of $\mathrm{Wr}(V)$ is real. Then V is disconjugate on every interval which contains no zero of $\mathrm{Wr}(V)$.*

Equivalently, if $I \subset \mathbb{R}$ and

$$\mathrm{Wr}(V)(x) \neq 0 \qquad \text{for every } x \in I,$$

then every non-zero $f \in V$ has at most $d-1$ zeros in I, counted with multiplicity.

For a simple example, consider

$$V_+ = \{a(x^2+1) + bx : a, b \in \mathbb{R}\}.$$

This time

$$\mathrm{Wr}(V_+)(x) = \det \begin{pmatrix} x^2+1 & 2x \\ x & 1 \end{pmatrix} = 1 - x^2.$$

The zeros of the Wronskian are -1 and 1, so the theorem says that every non-zero polynomial in V_+ has at most one zero in each of the intervals

$$(-\infty, -1), \qquad (-1, 1), \qquad (1, \infty).$$

This can also be checked directly. If $a \neq 0$, the two roots of $a(x^2+1) + bx$ have product 1, and therefore cannot both belong to any one of these three intervals.

A related celebrated result of Mukhin, Tarasov, and Varchenko states that if a complex vector space of polynomials has a Wronskian with only real zeros, then it has a basis consisting of polynomials with real coefficients (Mukhin et al. 2009). The Karp–Purbhoo theorem goes further: once the space is real, it controls how much each polynomial in the space can oscillate between consecutive zeros of the Wronskian.

Karp and Purbhoo prove a stronger positivity theorem from which disconjugacy follows. After a suitable fractional-linear change of variable, the interval I may be moved into $(0, \infty)$ while all zeros of the Wronskian are moved into $(-\infty, 0)$. Writing the coefficients of a basis of V as the rows of a matrix, the maximal minors of this matrix are called the Plücker coordinates of V. The positivity theorem implies that these coordinates may all be chosen non-negative.

A classical variation-diminishing theorem of Gantmakher and Krein then implies that the sequence of coefficients of every polynomial $f \in V$ changes sign at most $d-1$ times. Descartes' rule of signs gives at most $d-1$ positive zeros, counted with multiplicity. Reversing the change of variable gives the required bound on the original interval I.

The proof uses representation theory of symmetric groups, symmetric functions, and the KP hierarchy. It is striking that these advanced tools lead to such an elementary-looking conclusion: when the zeros of the Wronskian are all real, they are the only places at which the polynomials in V can acquire unusually many zeros.

References

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