A three-point connectivity constant for critical percolation

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The paper by Morris Ang, Gefei Cai, Xin Sun, and Baojun Wu (Ang et al. 2021), first posted online in 2021, has now appeared electronically in the *Journal of the American Mathematical Society*. The paper proves several exact formulas for conformal loop ensembles. One particularly concrete consequence is a rigorous proof of a remarkable prediction from mathematical physics: for critical site percolation on the triangular grid, a suitably normalized probability that three points belong to the same cluster converges to the universal constant

$$1.022013\ldots.$$

In this post we discuss critical Bernoulli site percolation on the triangular grid. It is convenient to identify the Euclidean plane with the complex plane $\mathbb{C}$, where $\mathrm{i}^2 = -1$, and to write

$$\mathbb{T} = \left\{ k + \frac{m}{2} + \mathrm{i}\frac{\sqrt{3}m}{2} \,\middle|\, k, m \in \mathbb{Z} \right\}.$$

Two vertices of $\mathbb{T}$ are joined by an edge if and only if their Euclidean distance is 1. For $\delta > 0$, let

$$\delta\mathbb{T} = \{\delta z : z \in \mathbb{T}\}$$

be the triangular grid with mesh size δ.

In Bernoulli site percolation, every vertex is independently declared *open* with some probability p and *closed* with probability $1 - p$. An *open path* is a path all of whose vertices are open, and an *open cluster* is a connected component of the subgraph induced by the open vertices.

For site percolation on the triangular grid, the critical probability is

$$p_{\mathrm{c}} = \frac{1}{2}.$$

We therefore declare every vertex of $\delta\mathbb{T}$ open or closed independently, each with probability $1/2$. At this critical value there is almost surely no infinite open cluster, but large clusters occur with probabilities governed by power laws.

For each $z \in \mathbb{C}$, choose a vertex $z^\delta \in \delta\mathbb{T}$ nearest to z. If there is more than one nearest vertex, choose one according to any fixed rule. Let $z_1, z_2, z_3 \in \mathbb{C}$ be pairwise distinct. For $1 \le i < j \le 3$, define

$$p_{ij}^\delta = \mathbb{P}\big(z_i^\delta \text{ and } z_j^\delta \text{ belong to the same open cluster}\big),$$

and define

$$p_{123}^\delta = \mathbb{P}\big(z_1^\delta, z_2^\delta, z_3^\delta \text{ belong to the same open cluster}\big).$$

As $\delta \to 0$, the lattice distance between any two of the vertices z_i^δ tends to infinity. Consequently,

$$p_{ij}^\delta \longrightarrow 0 \qquad \text{and} \qquad p_{123}^\delta \longrightarrow 0.$$

The question is whether these probabilities vanish in a related way.

The *normalized three-point connectivity* is

$$R_\delta(z_1, z_2, z_3) = \frac{p_{123}^\delta}{\sqrt{p_{12}^\delta p_{13}^\delta p_{23}^\delta}}.$$

The form of this normalization is not arbitrary. Heuristically, requiring a prescribed point to belong to a macroscopic cluster contributes one small “one-arm” factor. The probability p_{123}^δ contains three such factors. Each pairwise probability p_{ij}^δ contains two, and hence the square root of the product of the three pairwise probabilities again contains a total of three. The local small-scale factors should therefore cancel in the quotient.

This heuristic can be made precise. Let

$$\pi_\delta = \mathbb{P}\big(0 \text{ is joined by an open path to } \delta\mathbb{T} \setminus B(0,1)\big)$$

be the one-arm probability from scale δ to scale 1. Results on the scaling limit of critical percolation imply that the normalized limits

$$P_2(z_i, z_j) = \lim_{\delta \to 0} \frac{p_{ij}^\delta}{\pi_\delta^2}$$

and

$$P_3(z_1, z_2, z_3) = \lim_{\delta \to 0} \frac{p_{123}^\delta}{\pi_\delta^3}$$

exist and are conformally covariant.

Conformal covariance determines almost all of the dependence of these functions on the marked points. In fact, there are positive constants C_2 and C_3 such that

$$P_2(z_1, z_2) = C_2 |z_1 - z_2|^{-5/24}$$

and

$$P_3(z_1, z_2, z_3) = C_3 \prod_{1 \le i < j \le 3} |z_i - z_j|^{-5/48}.$$

Here $5/48$ is the one-arm exponent for critical percolation.

It follows that

$$\frac{P_3(z_1, z_2, z_3)}{\sqrt{P_2(z_1, z_2) P_2(z_1, z_3) P_2(z_2, z_3)}} = \frac{C_3}{C_2^{3/2}},$$

so all dependence on z_1, z_2, z_3 cancels. This is also natural from the viewpoint of conformal geometry: unlike a configuration of four points, a configuration of three points has no conformally invariant cross-ratio.

Conformal covariance does not, however, determine the number

$$\frac{C_3}{C_2^{3/2}}.$$

In the language of conformal field theory, this number is a *structure constant*. Two-point functions determine the normalization of the relevant fields, while the normalized three-point function contains additional information about the theory.

In 2010, Delfino and Viti used methods from conformal field theory to predict an exact value for this structure constant. Their answer is expressed through the *imaginary DOZZ formula*, an explicit special-function formula related to the three-point functions of Liouville conformal field theory.

Let C_b^{ImDOZZ} denote the imaginary DOZZ structure constant, and put

$$b = \frac{2}{\sqrt{6}} = \sqrt{\frac{2}{3}}, \qquad \alpha = \frac{1}{4b} - \frac{b}{2} = -\frac{1}{4\sqrt{6}}.$$

The Delfino–Viti prediction is

$$\mathcal{R}_{\mathrm{perc}} = \sqrt{2}\, C_b^{\mathrm{ImDOZZ}}(\alpha, \alpha, \alpha) = 1.022013\ldots.$$

The full definition of C_b^{ImDOZZ} involves the Barnes double gamma function and is too long to reproduce here; see (Ang et al. 2021) for the explicit formula.

Ang, Cai, Sun, and Wu proved that this prediction is correct.

Theorem 1 (Ang–Cai–Sun–Wu) *For every three pairwise distinct points* $z_1, z_2, z_3 \in \mathbb{C}$*,*

$$\lim_{\delta \to 0} R_\delta(z_1, z_2, z_3) = \mathcal{R}_{\mathrm{perc}},$$

where

$$\mathcal{R}_{\mathrm{perc}} = \sqrt{2}\, C^{\mathrm{ImDOZZ}}_{\sqrt{2/3}} \left(-\frac{1}{4\sqrt{6}}, -\frac{1}{4\sqrt{6}}, -\frac{1}{4\sqrt{6}} \right) = 1.022013\ldots.$$

In particular, the limit is independent of the positions of z_1, z_2, z_3*.*

Thus, after normalization by the three pairwise connection probabilities, all dependence on the size and shape of the triangle with vertices z_1, z_2, z_3 disappears. The same constant is obtained for an equilateral triangle, a very thin triangle, or three points separated by vastly different distances, provided that the points remain fixed and distinct as $\delta \to 0$.

If the three-point probability satisfied the naive factorization

$$p^\delta_{123} \approx \sqrt{p^\delta_{12} p^\delta_{13} p^\delta_{23}},$$

then the limiting constant would be 1. The actual value is approximately 2.2% larger. This small but nonzero correction measures a genuine three-way correlation that cannot be recovered from the three pairwise connection probabilities alone.

The scope of (Ang et al. 2021) goes well beyond the percolation constant. The same framework proves a conjecture of Ikhlef, Jacobsen, and Saleur concerning the nesting statistics of conformal loop ensembles and computes another three-point function associated with three points lying on the same loop. The paper therefore provides not only the value of one universal constant, but also a general method for deriving exact formulas for geometric observables in two-dimensional critical models.

References

Ang, Morris, Gefei Cai, Xin Sun, and Baojun Wu. 2021. "Integrability of Conformal Loop Ensemble: Imaginary DOZZ Formula and Beyond." *arXiv Preprint arXiv:2107.01788*.