Erdős' distinct distances problem in dimension 3

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In a recent post, I discussed the disproof of Erdős' unit distance conjecture. That problem asks how many pairs of points can be separated by one prescribed distance. A closely related problem asks the opposite kind of question: how few different distances can occur among all pairs of points?

On August 14, 2026, Jonathan Tidor, Hung-Hsun Hans Yu, and Dmitrii Zakharov posted a 147-page preprint resolving this problem in three-dimensional space, up to a subpolynomial factor (Tidor et al. 2026). Their result determines the correct exponent and shows that the three-dimensional integer lattice is essentially optimal.

Let P be a finite set of points in $\mathbb{R}^d$. Its *distance set* is

$$\Delta(P) = \big\{ \|p - q\| : p, q \in P,\ p \neq q \big\}.$$

Thus $|\Delta(P)|$ is the number of distinct positive distances determined by pairs of points of P. Define

$$D_d(N) = \min_{\substack{P \subset \mathbb{R}^d \\ |P| = N}} |\Delta(P)|.$$

The Erdős distinct distances problem asks for the asymptotic growth of $D_d(N)$ as N tends to infinity.

The natural example in $\mathbb{R}^3$ is a cubic section of the integer lattice. Suppose first that $N = m^3$ and let

$$P_m = \{1, \ldots, m\}^3.$$

The square of the distance between two points of P_m has the form

$$a^2 + b^2 + c^2, \qquad |a|, |b|, |c| \leq m - 1.$$

It is therefore an integer between 1 and $3(m-1)^2$. Consequently,

$$|\Delta(P_m)| \leq 3(m-1)^2 \ll m^2 = N^{2/3}.$$

For an arbitrary N, one can take N points from a cube of side length $\lceil N^{1/3} \rceil$, giving

$$D_3(N) \ll N^{2/3}.$$

In his original 1946 paper, Erdős conjectured that lattice constructions of this kind are asymptotically optimal (Erdős 1946). In the plane, the integer lattice determines approximately $N/\sqrt{\log N}$ distances. The celebrated theorem of Guth and Katz states that every set of N points in the plane determines at least a constant multiple of

$$\frac{N}{\log N}$$

distinct distances (Guth and Katz 2015). Thus the planar problem is solved up to a factor of order $\sqrt{\log N}$.

The higher-dimensional problem proved much more resistant. Before the work of Tidor, Yu, and Zakharov, the best lower bound in $\mathbb{R}^3$ followed by combining the Guth–Katz theorem with a bootstrapping argument of Solymosi and Vu (Solymosi and Vu 2008). It gave

$$D_3(N) \gg \frac{N^{3/5}}{(\log N)^{2/5}}.$$

There was therefore a substantial gap between the exponent $3/5$ in the best lower bound and the exponent $2/3$ supplied by the integer lattice.

The new theorem closes this gap.

Theorem 1 (Tidor–Yu–Zakharov) *There is a function $\varepsilon(N)$ satisfying*

$$\varepsilon(N) \ll \sqrt{\frac{\log \log N}{\log N}}$$

such that every set of N points in $\mathbb{R}^3$ determines at least

$$N^{2/3-\varepsilon(N)}$$

distinct distances.

Since $\varepsilon(N)$ tends to zero, the theorem and the lattice construction together give

$$D_3(N) = N^{2/3+o(1)}.$$

Equivalently, the lower bound can be written in the form

$$D_3(N) \geq N^{2/3} \exp\bigl(-O(\sqrt{\log N \log \log N})\bigr).$$

The factor multiplying $N^{2/3}$ is subpolynomial: it is smaller than a fixed power of N, but larger than the reciprocal of every fixed power of N. Thus the exponent $2/3$ is now known to be optimal. The result does not yet prove the stronger estimate

$$D_3(N) \gg N^{2/3}$$

with an absolute implied constant.

The contrast with the unit distance problem is striking. For unit distances, the long-standing expectation that a lattice should be extremal was recently disproved. For distinct distances in $\mathbb{R}^3$, the lattice heuristic survives: the exponent supplied by the cubic lattice is now known to be the correct one.

Two natural questions remain. The first is whether the subpolynomial loss can be eliminated, giving

$$D_3(N) \gg N^{2/3}.$$

The second is what happens in dimensions $d \geq 4$. The d-dimensional integer lattice gives

$$D_d(N) \ll N^{2/d},$$

but the corresponding lower bound is still unknown. Some components of the new proof may extend to higher dimensions, but geometric triality is a feature particular to three-dimensional rigid motions. Thus the resolution of the three-dimensional problem is both the end of a long-standing question and the beginning of a new set of incidence-geometric problems.

References

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