The Calabi–Yau conjectures for minimal surfaces

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This post continues my series on favorite theorems of the twenty-first century. For an overview of the categories and my earlier selections, see this introductory post.

My choice for 2004 in Geometry and Topology is the theorem of Colding and Minicozzi proving the Calabi–Yau conjectures for embedded minimal surfaces of finite topology. They showed that every such surface is properly embedded in $\mathbb{R}^3$ and that the only one contained in a halfspace is a plane. Their paper was received by the *Annals of Mathematics* in 2004 and published in the 2008 volume of the journal (Colding and Minicozzi 2008).

A smooth surface $M \subset \mathbb{R}^3$ is called *minimal* if its mean curvature vanishes identically. Equivalently, it is a critical point of the area functional under compactly supported deformations. The word *complete* refers to the intrinsic geometry of the surface: the Riemannian metric induced on M by the Euclidean metric of $\mathbb{R}^3$ is complete. Thus, a path that escapes every compact subset of M must have infinite length.

Completeness is an intrinsic condition and, by itself, says surprisingly little about how the surface sits in space. A surface can travel an infinite intrinsic distance while remaining confined to a bounded region of $\mathbb{R}^3$. This distinction lies at the heart of the Calabi–Yau conjectures.

In 1965, Calabi (Calabi 1965) formulated two conjectures about complete minimal surfaces in $\mathbb{R}^3$:

1. every complete minimal surface in $\mathbb{R}^3$ is unbounded;
2. every complete nonplanar minimal surface in $\mathbb{R}^3$ has an unbounded orthogonal projection onto every line.

The second statement is stronger than the first. For example, if a surface lies between two parallel planes, then its projection onto the line perpendicular to those planes is bounded.

If self-intersections are allowed, both conjectures are false. In 1980, Jorge and Xavier (Jorge and Xavier 1980) constructed a complete nonplanar minimal immersion contained between two parallel planes, disproving part (b) in the immersed setting. In 1996, Nadirashvili (Nadirashvili 1996) constructed a complete minimal immersion of a disk into a bounded ball in $\mathbb{R}^3$, disproving part (a).

These examples are immersions rather than embeddings: distinct points of the abstract surface may be mapped to the same point of $\mathbb{R}^3$. In 2000, Yau (Yau 2000) asked whether such counterexamples could be embedded, that is, realized without self-intersections.

To state the answer, we need two further notions. An embedded surface $M \subset \mathbb{R}^3$ is *properly embedded* if the inclusion map

$$M \longrightarrow \mathbb{R}^3$$

is proper: for every compact set $K \subset \mathbb{R}^3$, the intersection $M \cap K$ is compact in M. Equivalently, a sequence of points that escapes every compact subset of M must also escape every bounded subset of $\mathbb{R}^3$. Thus, properness rules out the possibility that the surface accumulates inside a bounded region while going to infinity intrinsically.

A connected surface M has *finite topology* if it is homeomorphic to

$$\widehat{M} \setminus \{p_1, \ldots, p_j\},$$

where $\widehat{M}$ is a compact surface without boundary. The genus of $\widehat{M}$ is the genus of M, while the removed points correspond to the *ends* of M. Intuitively, an end is a topologically distinct way of going to infinity along the surface. More formally, it is represented by a nested sequence of noncompact connected components of complements of increasingly large compact subsets of M. Thus, a surface has finite topology precisely when it has finite genus and finitely many ends.

Colding and Minicozzi proved the following theorem.

Theorem 1 (Colding–Minicozzi) *Let $M \subset \mathbb{R}^3$ be a connected, complete embedded minimal surface with finite topology. Then:*

(i) if M is contained in a halfspace of $\mathbb{R}^3$, then M is a plane;

(ii) M is properly embedded in $\mathbb{R}^3$.

Part (i) proves both of Calabi's conjectures for embedded surfaces of finite topology. Indeed, suppose first that M were bounded. It would then be contained in a halfspace, so part (i) would imply that M is a plane, which is impossible because a plane is unbounded.

Similarly, suppose that the orthogonal projection of a nonplanar surface M onto some line were bounded. The surface would then lie between two planes perpendicular to that line and, in particular, would be contained in a halfspace. Part (i) would again imply that M is a plane, a contradiction. In fact, the conclusion is slightly stronger than Calabi's conjecture: for every nonplanar M satisfying the hypotheses of the theorem, every linear coordinate function on M is unbounded both above and below.

Part (ii) is the deeper structural statement. Completeness alone allows a surface to travel infinitely far intrinsically while remaining in a compact part of the ambient space. Colding and Minicozzi proved that embeddedness and finite topology prevent this behavior.

The main geometric ingredient in their proof is a *chord–arc estimate* for embedded minimal disks. Euclidean distance between two points of a surface is always at most their intrinsic distance along the surface. Colding and Minicozzi established, after a suitable normalization, a converse estimate controlling intrinsic distance in terms of Euclidean distance. Roughly speaking, an embedded minimal disk cannot contain points that are extrinsically close but arbitrarily far apart intrinsically.

It follows that a sequence going to infinity intrinsically on a complete embedded minimal disk must also go to infinity in $\mathbb{R}^3$, proving properness for disks. To pass from disks to surfaces of finite topology, Colding and Minicozzi observe that, outside a compact core, such a surface is a finite union of annular ends. Far enough along each annular end, sufficiently large intrinsic balls are disks, so the disk estimates can be applied. They then prove that every annular end is proper, and hence that the entire surface is proper.

Once part (ii) is known, part (i) also follows from the strong halfspace theorem of Hoffman and Meeks (Hoffman and Meeks 1990), which states that a complete properly immersed minimal surface contained in a halfspace must be a plane.

Properness is an important regularity condition in the global theory of minimal surfaces, and many classical theorems were originally proved only under the assumption that the surface is properly embedded. Theorem 1 allows this hypothesis to be removed whenever the surface is complete, embedded, and has finite topology.

Another striking consequence is the classification of complete embedded minimal planar domains of finite topology. A planar domain is a surface of genus zero. Combining Theorem 1 with the classification results for properly embedded minimal surfaces shows that the only possibilities are

$$\text{the plane,} \qquad \text{the helicoid,} \qquad \text{and the catenoid.}$$

A natural conjecture is that the finite-topology assumption in Theorem 1 can be replaced by the weaker assumption of finite genus. More precisely, it is conjectured that every connected, complete embedded minimal surface

$$M \subset \mathbb{R}^3$$

with compact, possibly empty, boundary and finite genus is proper, with no restriction on the number of ends (Meeks et al. 2021).

Meeks, Pérez, and Ros (Meeks et al. 2021) proved this conjecture when the set of ends is countable. In the opposite direction, Collin, Kusner, Meeks, and Rosenberg (Collin et al. 2004) proved that every properly embedded minimal surface in $\mathbb{R}^3$ has at most two limit ends and, consequently, only countably many ends.

Thus, the remaining finite-genus case concerns surfaces with uncountably many ends. Any complete embedded minimal surface of finite genus with uncountably many ends would necessarily be nonproper. Whether such a surface exists remains an important open problem.

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