A new sphere-packing exponent

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On August 1, 2026, OpenAI released *Ten Advances in Mathematics and Theoretical Computer Science*, a collection of ten results in mathematics and theoretical computer science obtained by an internal OpenAI model. OpenAI states that the release also includes detailed reasoning walkthroughs and Lean certificates for the arguments. My personal favorite is the first chapter, entitled *Exponential Growth Rate of the Cohn–Elkies Sphere Packing Linear Program*. It determines the exact asymptotic strength of the Cohn–Elkies method in high dimensions and, as a consequence, gives the first improvement since 1978 to the best general exponential upper bound for sphere-packing density.

We have discussed sphere packing several times in this blog. Recall that a *packing of congruent balls*, or a *sphere packing*, in $\mathbb{R}^d$ is a family

$$\mathcal{P} = \{B(x_i, r) : i \in I\}$$

of pairwise disjoint open balls of the same radius. Let

$$P = \bigcup_{i \in I} B(x_i, r)$$

be the region covered by the packing. Its *upper density* is defined by

$$\overline{\delta}(\mathcal{P}) = \limsup_{R \to \infty} \frac{|P \cap B(x_0, R)|}{|B(x_0, R)|}, \tag{1}$$

where $|\cdot|$ denotes d-dimensional volume. The value in (1) does not depend on the choice of $x_0 \in \mathbb{R}^d$. If the limit exists, rather than merely the limit superior, it is called the *density* of the packing.

Let

$$\rho_d^* = \sup_{\mathcal{P}} \overline{\delta}(\mathcal{P})$$

denote the supremal sphere-packing density in $\mathbb{R}^d$. The radius of the balls is irrelevant, since a uniform rescaling preserves density. The exact value of ρ_d^* is known only in dimensions

$$d = 1, 2, 3, 8, \text{ and } 24.$$

In all other dimensions, and especially as $d \to \infty$, the problem remains remarkably mysterious.

In this previous post, we discussed recent progress on lower bounds for ρ_d^*. Klartag proved that there is a universal constant $c > 0$ such that

$$\rho_d^* \geq cd^2 2^{-d}$$

in every dimension. A subsequent result of Abuya, Gargava, and Zhao gives the stronger estimate

$$\rho_d^* \geq cd^2 \log\log d \, 2^{-d}$$

along an infinite sequence of dimensions. These polynomial improvements are substantial, but they do not change the exponential rate: after taking dth roots, both lower bounds tend to $1/2$.

For nearly half a century, the best general exponential upper bound was the 1978 estimate of Kabatianskii and Levenshtein (Kabatyanskiı and Levenshteın 1978):

$$\rho_d^* \leq 2^{-(\alpha_{\mathrm{KL}} + o(1))d}, \qquad \alpha_{\mathrm{KL}} = 0.59905576\ldots. \tag{2}$$

Later work improved lower-order factors in this estimate, but not the coefficient of d in the exponent.

In this earlier post, we discussed the Fourier-analytic method introduced by Cohn and Elkies. Because density is invariant under scaling, we may normalize the balls to have radius $1/2$. Thus, the distance between any two distinct centers is at least 1.

For an integrable function $f : \mathbb{R}^d \to \mathbb{R}$, write

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} \, dx$$

for its Fourier transform, and let

$$\omega_d = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2} + 1\right)}$$

be the volume of the unit ball in $\mathbb{R}^d$.

Suppose that f is a real-valued Schwartz function—that is, a smooth function that, together with all its derivatives, decays rapidly at infinity—and that

$$\widehat{f}(0) > 0, \qquad \widehat{f}(\xi) \geq 0 \quad \text{for every } \xi \in \mathbb{R}^d,$$

and

$$f(x) \leq 0 \quad \text{whenever } |x| \geq 1.$$

The Cohn–Elkies theorem then gives

$$\rho_d^* \le \frac{\omega_d}{2^d}\frac{f(0)}{\widehat{f}(0)}.$$

The idea is to sum f over differences between packing centers. The sign condition on f controls this sum in physical space, while the nonnegativity of $\widehat{f}$ controls the same quantity in Fourier space. For periodic packings, Poisson summation compares the two expressions; a limiting argument then yields the bound for arbitrary packings.

Let $\mathcal{A}_d$ be the class of functions satisfying the conditions above, and define

$$\mathrm{LP}_d = \frac{\omega_d}{2^d}\inf_{f\in\mathcal{A}_d}\frac{f(0)}{\widehat{f}(0)}.$$

Thus, LP_d is the best upper bound obtainable from the Cohn–Elkies method, and

$$\rho_d^* \le \mathrm{LP}_d\,.$$

Finding a strong sphere-packing bound therefore becomes an infinite-dimensional optimization problem: one must construct an auxiliary function with the required signs and make the ratio

$$\frac{f(0)}{\widehat{f}(0)}$$

as small as possible.

This method led to the exact solutions of the sphere-packing problem in dimensions 8 and 24. In those dimensions, Viazovska and, subsequently, Cohn, Kumar, Miller, Radchenko, and Viazovska constructed remarkable “magic functions” for which the Cohn–Elkies upper bound agrees with the density of the E_8 and Leech lattice packings.

In high dimensions, however, the optimal value of the linear program remained mysterious. It was known that the Cohn–Elkies method recovers the Kabatianskii–Levenshtein exponential rate, but it was not known whether optimizing over all admissible functions would produce a strictly better exponent.

The first chapter of the OpenAI report (OpenAI 2026) resolves this question completely.

Theorem 1 *As $d \to \infty$,*

$$\mathrm{LP}_d^{1/d} \longrightarrow \sqrt{\frac{e}{2\pi}}.$$

Consequently,

$$\limsup_{d\to\infty} (\rho_d^*)^{1/d} \leq \sqrt{\frac{e}{2\pi}}.$$

Since

$$\sqrt{\frac{e}{2\pi}} = 2^{-\alpha_*},$$

where

$$\alpha_* = \frac{1}{2}\log_2\left(\frac{2\pi}{e}\right) = 0.60440054\ldots,$$

Theorem 1 gives the new general upper bound

$$\rho_d^* \leq 2^{-(\alpha_*+o(1))d}. \tag{3}$$

Because a larger value of the exponent gives a smaller upper bound, (3) improves (2). The numerical change from

$$0.59905576\ldots \quad \text{to} \quad 0.60440054\ldots$$

may look modest, but the improvement is exponential: the new bound is smaller than the old one by a factor of

$$2^{-(0.00534478\ldots+o(1))d}.$$

The word "exact" in Theorem 1 refers to the strength of the Cohn–Elkies method, not to the true value of ρ_d^*. The true high-dimensional sphere-packing density remains far from understood. Combining the best general lower bound with Theorem 1 gives

$$\frac{1}{2} \leq \liminf_{d\to\infty}(\rho_d^*)^{1/d} \leq \limsup_{d\to\infty}(\rho_d^*)^{1/d} \leq \sqrt{\frac{e}{2\pi}} = 0.65774462\ldots.$$

Thus, an exponential gap remains. Nevertheless, the new theorem identifies the precise frontier of one of the most powerful general methods in discrete geometry and, at the same time, improves a high-dimensional sphere-packing exponent that had stood unchanged for almost fifty years.

References

Kabatyanskiı, Grigorii Anatol'evich, and Vladimir Iosifovich Levenshteın. 1978. "On Bounds for Packings on a Sphere and in Space." *Problemy Peredachi Informatsii* 14 (1): 3–25.

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