Polya’s conjecture is true for balls

Bogdan Grechuk • 9 Aug 2026

On July 31, 2026, Nikolay Filonov, Michael Levitin, Iosif Polterovich, and David A. Sher posted the preprint *Pólya’s conjecture for higher-dimensional Neumann balls* (Filonov et al. 2026). In it, they prove the remaining Neumann case of Pólya’s conjecture for Euclidean balls in dimensions $d \geq 3$. Combined with their earlier work on disks and Dirichlet eigenvalues (Filonov et al. 2023), this establishes Pólya’s conjecture for Euclidean balls in every dimension $d \geq 2$.

We have previously discussed the hot spots conjecture in this post and this follow-up post. That conjecture concerns the eigenfunctions associated with the first nonzero Neumann eigenvalue of a planar domain. Pólya’s conjecture addresses a different and more global question: can every Dirichlet and Neumann eigenvalue be bounded solely in terms of its position in the spectrum and the volume of the domain?

Let $d \geq 2$, and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Recall that the Laplacian is the differential operator

$$\Delta u = \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2}.$$

The Dirichlet eigenvalue problem asks for nonzero functions u satisfying

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Its eigenvalues form a discrete sequence

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_n(\Omega) \leq \cdots, \qquad \lambda_n(\Omega) \longrightarrow \infty,$$

where every eigenvalue is repeated according to its multiplicity.

The corresponding Neumann eigenvalue problem is

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where ν is the outward-pointing unit normal vector on $\partial\Omega$. Its eigenvalues can similarly be listed as

$$0 = \mu_1(\Omega) < \mu_2(\Omega) \leq \cdots \leq \mu_n(\Omega) \leq \cdots, \qquad \mu_n(\Omega) \longrightarrow \infty.$$

The first eigenvalue is zero because every constant function is a Neumann eigenfunction.

Weyl's law describes the asymptotic growth of both sequences. If $|\Omega|$ denotes the volume of Ω and

$$\omega_d = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}$$

is the volume of the unit ball in $\mathbb{R}^d$, then

$$\lambda_n(\Omega) \sim \mu_n(\Omega) \sim 4\pi^2 \left(\frac{n}{\omega_d |\Omega|}\right)^{2/d} \qquad \text{as } n \to \infty.$$

Thus, to leading order, the high eigenvalues depend only on the dimension and the volume of the domain.

In 1954, Pólya conjectured that the leading term in Weyl's law gives one-sided bounds for every eigenvalue, not merely an asymptotic approximation (Pólya 1954). More precisely, he predicted that

$$\mu_{n+1}(\Omega) \leq 4\pi^2 \left(\frac{n}{\omega_d |\Omega|}\right)^{2/d} \leq \lambda_n(\Omega), \qquad n = 1, 2, \ldots. \tag{1}$$

The left-hand inequality is called the Neumann Pólya conjecture, while the right-hand inequality is called the Dirichlet Pólya conjecture. In other words, Pólya predicted that the Weyl approximation always lies between the corresponding Neumann and Dirichlet eigenvalues.

Pólya proved his conjecture for domains that tile Euclidean space, meaning that congruent copies of the domain cover $\mathbb{R}^d$, up to a set of measure zero, without overlapping. A technical restriction in his original Neumann argument was subsequently removed by Kellner (Pólya 1961; Kellner 1966). This covers familiar examples such as rectangles and many polygons, but not disks or higher-dimensional balls.

For arbitrary domains, both inequalities in (1) are known in full generality for $n = 1$ and $n = 2$. The cases $n = 1$ follow from the Faber–Krahn and Szegő–Weinberger inequalities, while the Dirichlet inequality for $n = 2$ follows from the Krahn–Szegő inequality (Henrot 2006). In 2019, Bucur and Henrot proved the remaining Neumann inequality for $n = 2$ (Bucur and Henrot 2019).

A major breakthrough came in 2023, when Filonov, Levitin, Polterovich, and Sher proved both Pólya inequalities for the disk (Filonov et al. 2023). The disk thereby became the first non-tiling planar domain for which the full conjecture was established. In the same paper, they proved the Dirichlet inequality for Euclidean balls in every dimension. The Neumann problem for balls in dimensions $d \geq 3$, however, remained open.

The new preprint (Filonov et al. 2026) resolves precisely this remaining case. Together, the two papers give the following theorem.

Theorem 1 *Let $B \subset \mathbb{R}^d$ be a Euclidean ball, where $d \geq 2$. Then, for every $n \geq 1$,*

$$\mu_{n+1}(B) \leq 4\pi^2 \left(\frac{n}{\omega_d |B|}\right)^{2/d} \leq \lambda_n(B).$$

Thus, both the Dirichlet and Neumann Pólya conjectures hold for Euclidean balls in every dimension.

Although the eigenvalues of a ball can be described explicitly using Bessel functions, this does not make Pólya's inequalities easy to prove. In the higher-dimensional Neumann problem, the eigenvalues are determined by zeros of derivatives of ultraspherical Bessel functions and occur with dimension-dependent multiplicities. Consequently, the spectrum is assembled from many interlaced families of numbers, making direct ordering essentially impossible.

The proof in (Filonov et al. 2026) combines several different ideas. The high-frequency range is treated using bounds for Bessel phase functions and weighted lattice-point estimates. Low-lying eigenvalues are controlled through variational arguments involving carefully chosen dimension-dependent test functions. The finitely many parameter ranges left between these two regimes are handled by a rigorous computer-assisted verification using exact rational arithmetic and certified enclosures for the relevant zeros. Together, these ingredients complete the proof of Pólya's conjecture for one of the most natural families of domains in spectral geometry.

References

Bucur, Dorin, and Antoine Henrot. 2019. "Maximization of the Second Non-Trivial Neumann Eigenvalue." *Acta Math.* 222 (2): 337–61.

Filonov, Nikolay, Michael Levitin, Iosif Polterovich, and David A. Sher. 2023. "Pólya's Conjecture for Euclidean Balls." *Invent. Math.* 234 (1): 129–69. https://doi.org/10.1007/s00222-023-01198-1.

Filonov, Nikolay, Michael Levitin, Iosif Polterovich, and David A. Sher. 2026. "Pólya's Conjecture for Higher-Dimensional Neumann Balls." *arXiv Preprint arXiv:2607.29305*.

Henrot, Antoine. 2006. *Extremum Problems for Eigenvalues of Elliptic Operators*. Frontiers in Mathematics. Birkhäuser Verlag, Basel.

Kellner, R. 1966. “On a Theorem of Polya.” *Amer. Math. Monthly* 73: 856–58. https://doi.org/10.2307/2314181.

Pólya, G. 1961. “On the Eigenvalues of Vibrating Membranes.” *Proc. London Math. Soc. (3)* 11: 419–33. https://doi.org/10.1112/plms/s3-11.1.419.

Pólya, George. 1954. *Mathematics and plausible reasoning. Vol. 1: Induction and analogy in mathematics*. Princeton University Press.