Counting low-degree number fields

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This post continues my series on favorite theorems of the twenty-first century. For an overview of the categories and my earlier selections, see this introductory post.

My choice for 2004 in Algebra is Manjul Bhargava's paper *The Density of Discriminants of Quintic Rings and Fields*. The paper was received by the *Annals of Mathematics* in 2004 and appeared in print in 2010 (Bhargava 2010).

A *number field* is a finite extension F of $\mathbb{Q}$. Its dimension as a vector space over $\mathbb{Q}$ is called its *degree* and is denoted by

$$[F:\mathbb{Q}] = n.$$

For every $x \in F$, multiplication by x defines a $\mathbb{Q}$-linear map

$$m_x : F \longrightarrow F, \qquad m_x(y) = xy.$$

The trace of this linear transformation is called the *field trace* of x and is denoted by

$$\mathrm{Tr}_{F/\mathbb{Q}}(x).$$

Since the trace of a linear transformation is independent of the choice of basis, this definition is intrinsic to the field extension $F/\mathbb{Q}$.

An element $x \in F$ is called an *algebraic integer* if it is a root of a monic polynomial with integer coefficients. The algebraic integers in F form a ring, denoted by $\mathcal{O}_F$ and called the *ring of integers* of F. As an abelian group, $\mathcal{O}_F$ is free of rank n. A basis

$$e_1, e_2, \ldots, e_n$$

of $\mathcal{O}_F$ as a $\mathbb{Z}$-module is called an *integral basis* of F.

The *discriminant* of F is

$$\mathrm{Disc}(F) = \det\left(\mathrm{Tr}_{F/\mathbb{Q}}(e_i e_j)\right)_{1 \le i,j \le n}.$$

This is a nonzero integer that does not depend on the choice of integral basis.

For $X > 0$, let $N_n(X)$ denote the number of isomorphism classes of number fields F of degree n satisfying

$$|\mathrm{Disc}(F)| \leq X.$$

The Hermite–Minkowski theorem implies that $N_n(X)$ is finite for every fixed n and X. An old folklore conjecture predicts that, for every fixed $n \geq 2$, there is a positive constant c_n such that

$$\lim_{X \to \infty} \frac{N_n(X)}{X} = c_n > 0. \qquad (1)$$

Equivalently, the conjecture asserts that

$$N_n(X) \sim c_n X \qquad \text{as } X \to \infty.$$

The case $n = 2$ is elementary. In 1971, Davenport and Heilbronn (Davenport and Heilbronn 1971) proved the conjecture for cubic fields. In 2005, Bhargava (Bhargava 2005) established the next case, and proved that the number of quartic number fields satisfies

$$N_4(X) \sim c_4 X,$$

where

$$c_4 = 0.30580\ldots.$$

n the paper selected here, Bhargava proved the conjecture for quintic fields and obtained an explicit formula for the asymptotic constant.

Theorem 1 *The number of quintic number fields satisfies*

$$N_5(X) \sim c_5 X,$$

where

$$c_5 = \frac{13}{120} \prod_{p \in \mathcal{P}} \left(1 + p^{-2} - p^{-4} - p^{-5}\right) = 0.149674\ldots,$$

and $\mathcal{P}$ denotes the set of prime numbers.

The proofs for quartic and quintic fileds follow the same broad strategy. Bhargava's remarkable algebraic parametrizations encode quartic and quintic rings as integral orbits in concrete representation spaces. This transforms the problem of counting number fields into a problem of counting lattice points of bounded discriminant. Methods from the geometry of numbers provide the main asymptotic count, while a delicate sieve removes the nonmaximal orders and leaves precisely the rings of integers of number fields.

The quartic argument relies on Bhargava's parametrization of quartic rings, while the quintic argument uses his parametrization of quintic rings, discussed in this earlier post.

References

Bhargava, Manjul. 2005. “The Density of Discriminants of Quartic Rings and Fields.” *Ann. of Math.* 162 (2): 1031–63.

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