On the two-point logarithmic Chowla conjecture

Bogdan Grechuk • 5 Aug 2026

Cédric Pilatte's paper (Pilatte 2023), first posted online in 2023, has now been accepted for publication in the *Journal of the American Mathematical Society*. The paper proves the strongest known quantitative form of the two-point logarithmically averaged Chowla conjecture, replacing a saving involving $\log\log x$ by a fixed power of $\log x$.

The *Liouville function* is defined by

$$\lambda(n) = (-1)^{\Omega(n)},$$

where $\Omega(n)$ denotes the number of prime factors of the positive integer n, counted with multiplicity. Thus, $\lambda(n) = 1$ when n has an even number of prime factors and $\lambda(n) = -1$ when it has an odd number. For example,

$$\lambda(12) = \lambda(2^2 \cdot 3) = (-1)^3 = -1.$$

The function λ is completely multiplicative, meaning that

$$\lambda(mn) = \lambda(m)\lambda(n)$$

for all positive integers m and n.

Many central conjectures in analytic number theory express the principle that the values of the Liouville function should behave statistically like independent random signs. One of the most famous is the *Chowla conjecture* (Chowla 1965). Let $k \geq 1$, let $a_1, \ldots, a_k$ be positive integers, and let $b_1, \ldots, b_k$ be distinct non-negative integers satisfying

$$a_i b_j - a_j b_i \neq 0 \qquad \text{whenever } 1 \leq i < j \leq k.$$

The last condition ensures that no two of the linear forms $a_i n + b_i$ are proportional. The Chowla conjecture states that

$$\sum_{n \leq x} \lambda(a_1 n + b_1) \cdots \lambda(a_k n + b_k) = o(x) \qquad \text{as } x \to \infty. \tag{1}$$

This prediction is natural if one thinks of the values of λ as random signs. Under this heuristic, the product

$$\lambda(a_1 n + b_1) \cdots \lambda(a_k n + b_k)$$

should be equal to 1 for approximately half of the integers n and to -1 for approximately the other half, resulting in substantial cancellation.

The simplest case $k = 1$ includes the assertion

$$\sum_{n \le x} \lambda(n) = o(x),$$

which is equivalent to the prime number theorem. The first genuinely new case, $k = 2$, remains open. For example, the Chowla conjecture predicts that

$$\sum_{n \le x} \lambda(n)\lambda(n+1) = o(x).$$

In other words, the parity of the number of prime factors of n should be asymptotically uncorrelated with that of $n + 1$.

Because the original Chowla conjecture appears extremely difficult, Tao (Tao 2016) considered its *logarithmically averaged* version. This replaces the ordinary average by one in which the integer n is assigned weight $1/n$. The conjectural estimate becomes

$$\sum_{n \le x} \frac{\lambda(a_1 n + b_1) \cdots \lambda(a_k n + b_k)}{n} = o(\log x) \qquad \text{as } x \to \infty. \tag{2}$$

The estimate (2) follows from (1) by partial summation and is therefore formally weaker. Nevertheless, it retains many of the important consequences of the original conjecture and has proved considerably more accessible.

In 2017, Tao (Tao 2017) proved the two-point logarithmically averaged Chowla conjecture. Tao's theorem takes the following quantitative-uniform form.

Theorem 1 (Tao) *Let a_1, a_2 be positive integers, and let b_1, b_2 be integers satisfying*

$$a_1 b_2 - a_2 b_1 \neq 0.$$

Let $w = w(x)$ satisfy

$$1 \le w(x) \le x \qquad \textit{and} \qquad w(x) \to \infty$$

as $x \to \infty$. Then

$$\sum_{x/w(x) < n \le x} \frac{\lambda(a_1 n + b_1)\lambda(a_2 n + b_2)}{n} = o(\log w(x)) \qquad \textit{as } x \to \infty.$$

Taking

$$a_1 = a_2 = b_2 = 1 \qquad \text{and} \qquad b_1 = 0,$$

Theorem 1 gives

$$\frac{1}{\log w(x)} \sum_{x/w(x)<n\leq x} \frac{\lambda(n)\lambda(n+1)}{n} = o(1).$$

An explicit analysis of Tao's argument gives an upper bound of the form

$$O\left(\min\left\{\log\log\log w(x), \log\log\log\log x\right\}^{-1/5}\right)$$

for the $o(1)$ term.

In 2021, Helfgott and Radziwiłł (Helfgott and Radziwiłł 2021) obtained the much stronger bound

$$O\left(\frac{1}{\sqrt{\log\log w(x)}}\right).$$

Theorem 2 (Helfgott–Radziwiłł) *Let $w = w(x)$ satisfy*

$$e < w(x) \leq x \qquad \text{and} \qquad w(x) \to \infty$$

as $x \to \infty$. Then

$$\frac{1}{\log w(x)} \sum_{x/w(x)<n\leq x} \frac{\lambda(n)\lambda(n+1)}{n} = O\left(\frac{1}{\sqrt{\log\log w(x)}}\right) \qquad \text{as } x \to \infty. \tag{3}$$

In particular, taking $w(x) = x$ gives

$$\sum_{n\leq x} \frac{\lambda(n)\lambda(n+1)}{n} = O\left(\frac{\log x}{\sqrt{\log\log x}}\right).$$

The significance of the work of Helfgott and Radziwiłł lies not only in this quantitative improvement, but also in the method they introduced. In his proof of Theorem 1, Tao considered a divisibility graph associated with a collection of small primes. Its vertices are integers, and n is joined to $n \pm p$ whenever p is one of the chosen primes and p divides n. Notice that in this case p also divides $n \pm p$.

Tao observed that a suitable local expansion property for this graph would force the required cancellation in the Liouville correlations. Informally, expansion means that a short random walk beginning at a typical integer rapidly spreads among many nearby integers. One could then combine this spreading with the fact that the Liouville function has small average on most short intervals.

Tao was unable to establish the required expansion property and instead completed his proof using the entropy decrement method. Although successful, that argument produced a relatively weak rate of convergence and was difficult to adapt to related problems.

Helfgott and Radziwiłł succeeded in implementing Tao's original strategy. More precisely, they studied a weighted adjacency operator associated with the divisibility graph and proved that it exhibits strong local expansion after removing a sparse exceptional set of vertices. Combining this spectral result with estimates for the Liouville function on short intervals yielded Theorem 2, as well as several further applications.

There is, however, a natural limitation to the method in this form. The total weight contributed by steps of prime length is governed by a sum of reciprocals of primes and is therefore only of order $\log\log x$. Even an essentially optimal spectral estimate for this graph consequently produces a saving involving $\log\log x$, rather than a fixed power of $\log x$.

Pilatte's key idea is to replace steps of prime length by steps whose lengths are products of many primes. Instead of averaging only over primes p, he averages over integers of the form

$$d = p_1 p_2 \cdots p_J,$$

where J is of order $\log\log x$. This creates a far larger collection of admissible steps: their total reciprocal weight can be a fixed power of $\log x$. Pilatte then develops a substantially more intricate spectral analysis for the resulting weighted graph. Among the new ingredients are non-backtracking operators, a high-trace argument, and estimates for systems of interacting divisibility constraints.

This leads to a power saving in the logarithm.

Theorem 3 (Pilatte) *There exists an absolute constant $c > 0$ such that*

$$\sum_{n \le x} \frac{\lambda(n)\lambda(n+1)}{n} = O\left((\log x)^{1-c}\right) \qquad \text{as } x \to \infty.$$

Equivalently,

$$\frac{1}{\log x} \sum_{n \le x} \frac{\lambda(n)\lambda(n+1)}{n} = O\left((\log x)^{-c}\right).$$

Thus Pilatte replaces the square-root saving in $\log\log x$ from Theorem 2 by a fixed power saving in $\log x$. This is a dramatic quantitative improvement and appears to be the strongest estimate obtainable from the currently available short-interval methods.

References

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