Complete Intersections for Permutations

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In July 2026, Keller, Kupavskii, Lifshitz, and Sheinfeld posted a preprint (Keller et al. 2026) proving a complete-intersection theorem for permutations. Their result determines the largest families of permutations in which every pair agrees in many positions, provided only that the number of symbols is larger than an absolute constant.

A central theme in combinatorics is to determine how large a collection of objects can be when its members satisfy a prescribed intersection condition. A classical example is the *Erdős–Ko–Rado theorem*. Let

$$[n] = \{1, 2, \ldots, n\},$$

and let $\mathcal{F}$ be a family of r-element subsets of $[n]$ such that

$$A \cap B \neq \varnothing \qquad \text{for all } A, B \in \mathcal{F}.$$

Thus, $\mathcal{F}$ is a pairwise intersecting family. For any fixed $i \in [n]$, the family

$$\mathcal{F}_i = \{A \subseteq [n] : |A| = r \text{ and } i \in A\}$$

is pairwise intersecting and has size

$$|\mathcal{F}_i| = \binom{n-1}{r-1}.$$

In 1961, Erdős, Ko, and Rado (Erdős et al. 1961) proved that, whenever $n \geq 2r$,

$$|\mathcal{F}| \leq \binom{n-1}{r-1}.$$

If $n > 2r$, equality holds if and only if $\mathcal{F} = \mathcal{F}_i$ for some $i \in [n]$.

There is a higher-intersection version of this result. Suppose that $\mathcal{F}$ is a family of r-element subsets of $[n]$ satisfying

$$|A \cap B| \geq k \qquad \text{for all } A, B \in \mathcal{F}.$$

For fixed k and r, and for n sufficiently large in terms of these parameters,

$$|\mathcal{F}| \leq \binom{n-k}{r-k}.$$

Moreover, equality holds precisely when $\mathcal{F}$ consists of all r-element subsets containing some fixed k-element subset of $[n]$.

In 1977, Frankl and Deza (Frankl and Deza 1977) conjectured that an analogous phenomenon should hold for permutations. A permutation of $[n]$ is a bijection $\sigma\colon [n] \to [n]$, and S_n denotes the set of all such permutations. Two permutations $\sigma, \tau \in S_n$ are said to *k-intersect* if they agree in at least k positions; equivalently,

$$\bigl|\{i \in [n] : \sigma(i) = \tau(i)\}\bigr| \geq k.$$

A family $I \subseteq S_n$ is *k-intersecting* if every two permutations in I k-intersect.

Fix a permutation $\sigma \in S_n$ and distinct indices $i_1, \ldots, i_k \in [n]$. Consider the family

$$I(\sigma; i_1, \ldots, i_k) = \{\tau \in S_n : \tau(i_t) = \sigma(i_t) \text{ for every } t = 1, \ldots, k\}.$$

Every two members of this family agree at the positions $i_1, \ldots, i_k$, so the family is k-intersecting. Once the images of these k positions have been prescribed, the remaining $n-k$ values may be permuted arbitrarily. Consequently,

$$|I(\sigma; i_1, \ldots, i_k)| = (n-k)!.$$

Frankl and Deza conjectured that, for each fixed k and all sufficiently large n, this construction is optimal. Ellis, Friedgut, and Pilpel proved the conjecture in 2011 (Ellis et al. 2011). Their proof combines eigenvalue methods with representation theory.

Theorem 1 (Ellis–Friedgut–Pilpel) *For every positive integer k, there exists an integer $n_0(k)$ such that the following holds whenever $n \geq n_0(k)$. If $I \subseteq S_n$ is k-intersecting, then*

$$|I| \leq (n-k)!.$$

Equality holds if and only if

$$I = I(\sigma; i_1, \ldots, i_k)$$

for some $\sigma \in S_n$ and some distinct $i_1, \ldots, i_k \in [n]$.

Let $f(k,n)$ denote the maximum size of a k-intersecting subset of S_n. Theorem 1 shows that

$$f(k,n) = (n-k)!$$

when n is sufficiently large in terms of k. It does not, however, determine $f(k,n)$ when k is allowed to be large relative to n.

Ellis, Friedgut, and Pilpel proposed a family of constructions intended to describe the answer for every pair of integers $1 \leq k \leq n$. For an integer r satisfying

$$0 \leq r \leq \left\lfloor \frac{n-k}{2} \right\rfloor,$$

define

$$\mathcal{F}(n,k,r) = \left\{ \sigma \in S_n : \left| \{ i \in [k+2r] : \sigma(i) = i \} \right| \geq k+r \right\}.$$

In other words, a permutation belongs to $\mathcal{F}(n,k,r)$ if it fixes at least $k+r$ of the first $k+2r$ positions.

The family $\mathcal{F}(n,k,r)$ is k-intersecting. Indeed, if $\sigma, \tau \in \mathcal{F}(n,k,r)$, then each fixes at least $k+r$ positions in a set of size $k+2r$. Their sets of fixed positions therefore have an intersection of size at least

$$2(k+r) - (k+2r) = k.$$

At each of these common fixed positions, σ and τ agree. Hence

$$f(k,n) \geq \max_{0 \leq r \leq \lfloor (n-k)/2 \rfloor} |\mathcal{F}(n,k,r)|.$$

Ellis, Friedgut, and Pilpel conjectured that equality always holds. This statement is the permutation analogue of a complete-intersection theorem: depending on the relationship between k and n, the optimal family need not be obtained merely by prescribing the values of a permutation at k positions. Instead, one of the more general families $\mathcal{F}(n,k,r)$ may be larger.

Keller, Kupavskii, Lifshitz, and Sheinfeld (Keller et al. 2026) have now proved this conjecture for every sufficiently large n, with a threshold that is independent of k.

Theorem 2 (Keller–Kupavskii–Lifshitz–Sheinfeld) *There exists an absolute constant n_0 such that, for every $n \geq n_0$ and every integer k satisfying $1 \leq k \leq n$,*

$$f(k,n) = \max_{0 \leq r \leq \lfloor (n-k)/2 \rfloor} |\mathcal{F}(n,k,r)|.$$

Thus, once n exceeds a single absolute threshold, the maximum size of a k-intersecting family of permutations is obtained by one of the explicit complete-intersection constructions above, simultaneously for every possible value of k.

References

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