The Newtonian 4-body problem

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This post continues my series on favorite theorems from the twenty-first century. For an overview of the categories and my earlier selections, see this post.

My choice for 2004 in analysis is Hampton and Moeckel's proof that the planar Newtonian 4-body problem has only finitely many relative equilibria. Their paper was submitted in 2004 and published in 2006 (Hampton and Moeckel 2006).

Many important problems in physics reduce to systems of differential equations. One of the most famous is the n-body problem: describe the motion of n point particles interacting through gravity, either in space or in the plane.

Let the particles have masses $m_i > 0$ and positions $x_i \in \mathbb{R}^d$. After choosing units in which the gravitational constant is 1, Newton's equations take the form

$$m_j \frac{\mathrm{d}^2 x_j}{\mathrm{d}t^2} = \sum_{i \neq j} \frac{m_i m_j (x_i - x_j)}{r_{ij}^3}, \qquad 1 \leq j \leq n, \tag{1}$$

where $r_{ij} = \|x_i - x_j\|$ is the distance between the ith and jth particles. Even in the planar case $d = 2$, solutions of (1) can be extremely complicated. A natural first step is therefore to classify particularly simple motions, among them the relative equilibria.

In a relative equilibrium, the entire configuration rotates rigidly, with constant angular velocity, while retaining its shape. More precisely, a planar relative equilibrium is a solution of (1) of the form

$$x_i(t) = c + R_{\omega t}\big(x_i(0) - c\big),$$

where $c \in \mathbb{R}^2$ is the center of rotation, $\omega \neq 0$ is constant, and $R_{\omega t}$ denotes rotation through angle ωt. Two relative equilibria are regarded as equivalent when one can be obtained from the other by a translation, a rotation, and a dilation of the plane.

For a fixed number $n \geq 3$ of bodies and fixed positive masses, must there be only finitely many equivalence classes of relative equilibria? This question was posed by Chazy in 1918 (Chazy 1918) and later appeared as Problem 6 in Smale's list of mathematical problems for the twenty-first century (Smale 1998). Wintner made the stronger conjecture that the number

$$q = q(n, m_1, \ldots, m_n)$$

of equivalence classes is bounded above by a number q_n depending only on n, and not on the particular positive masses (Wintner 1941).

For $n = 3$, the answer has been known for centuries. For every choice of positive masses, there are five relative equilibria: two equilateral configurations, described by Lagrange in 1772 (Lagrange 1772), and three collinear configurations, discovered by Euler in 1767 (Euler 1767).

The next case is already much harder. In 1977, Kuz'mina proved finiteness for $n = 4$ and generic positive masses (Kuz'mina 1977). More precisely, there is a subset

$$A \subset \mathbb{R}_+^4$$

of four-dimensional Lebesgue measure zero such that, for every mass vector

$$m = (m_1, m_2, m_3, m_4) \in \mathbb{R}_+^4 \setminus A,$$

the planar Newtonian 4-body problem has only finitely many equivalence classes of relative equilibria. This left open the possibility that exceptional choices of positive masses might admit infinitely many.

Hampton and Moeckel eliminated that possibility.

Theorem 1 (Hampton–Moeckel) *For the planar Newtonian* 4*-body problem* (1)*, there are only finitely many equivalence classes of relative equilibria for every choice of positive masses* m_1, m_2, m_3, m_4*. In fact, their number is at most* 8472*.*

The proof replaces the original equations by a large collection of reduced systems that are easier to analyze individually. There are too many such systems for a practical hand calculation, but few enough to be handled by a computer. Crucially, the required computations are symbolic or use exact integer arithmetic, so the computer-assisted parts of the argument introduce no numerical approximation and the resulting proof is fully rigorous.

The bound 8472 is not expected to be sharp. Its importance lies instead in establishing a uniform finite upper bound valid for every choice of four positive masses. Thus Hampton and Moeckel proved not only the finiteness conjecture for $n = 4$, but also Wintner's stronger uniformity conjecture in this case.

Subsequent progress has been more limited. Albouy and Kaloshin proved that the planar 5-body problem has only finitely many relative equilibria for generic positive masses (Albouy and Kaloshin 2012). More precisely, there is an exceptional set

$$A \subset \mathbb{R}_+^5$$

of five-dimensional Lebesgue measure zero such that, whenever

$$m = (m_1, m_2, m_3, m_4, m_5) \in \mathbb{R}_+^5 \setminus A,$$

only finitely many equivalence classes occur. The exceptional set may be taken to be an explicit algebraic subvariety of the mass space of codimension two.

The positivity of the masses is essential. Roberts showed in 1999 that, if negative masses are allowed, the planar 5-body problem can possess a continuum of relative equilibria (Roberts 1999).

Albouy and Kaloshin's proof studies the possible singular behavior of complex solutions of the equations defining relative equilibria. For $n > 5$, however, the resulting singularities appear to become considerably more complicated, and the method encounters serious obstacles. Further progress may therefore require a substantially new idea.

References

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