On Falconer’s distance set problem on the plane

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This is the fourth post in a series begun here and continued here and here. The series presents one striking and accessible theorem associated with each of the 2026 Fields Medalists and the 2026 Abacus Medalist. This post concerns Fields Medalist Hong Wang.

A central achievement of Wang’s work is her resolution, together with Joshua Zahl, of the three-dimensional Kakeya conjecture. This conjecture states that every compact subset of $\mathbb{R}^3$ containing a unit line segment in every direction must have Hausdorff dimension 3. We have already discussed this result in a previous blog post. Here, instead, we turn to another famous problem to which Wang has made a major contribution: Falconer’s distance set problem.

For a set $E \subset \mathbb{R}^d$, define its distance set by

$$\Delta(E) = \big\{|x-y| : x, y \in E\big\}.$$

Thus, $\Delta(E)$ records all distances occurring between pairs of points of E. In 1985, Falconer (Falconer 1985) asked for the smallest threshold $c(d)$ with the following property: whenever a compact set $E \subset \mathbb{R}^d$ has Hausdorff dimension greater than $c(d)$, its distance set $\Delta(E)$ has positive Lebesgue measure.

Falconer constructed a lattice-based example of a compact set $E \subset \mathbb{R}^d$ with Hausdorff dimension $d/2$ whose distance set has Lebesgue measure zero. Consequently,

$$\frac{d}{2} \leq c(d).$$

It is widely conjectured that this example gives the optimal obstruction, so that

$$c(d) = \frac{d}{2}$$

for every $d \geq 2$. This statement is known as *Falconer’s distance conjecture*.

Falconer’s original argument also gave the upper bound

$$c(d) \leq \frac{d+1}{2}$$

for every $d \geq 2$. In 1999, Wolff (Wolff 1999) proved the substantially stronger planar estimate

$$c(2) \leq \frac{4}{3}.$$

In 2005, Erdoğan (Erdogan 2005) extended this type of estimate to higher dimensions, proving that

$$c(d) \leq \frac{d}{2} + \frac{1}{3}$$

for every $d \geq 2$.

Further progress came in 2018, when Du, Guth, Ou, Wang, Wilson, and Zhang (Du, Guth, et al. 2021) proved that

$$c(3) \leq \frac{9}{5}$$

and, for every $d \geq 4$,

$$c(d) \leq \frac{d}{2} + \frac{1}{4} + \frac{d+1}{4(2d+1)(d-1)}.$$

In 2019, Du and Zhang (Du and Zhang 2019) established the more uniform bound

$$c(d) \leq \frac{d^2}{2d-1}.$$

This recovers the bounds $c(2) \leq 4/3$ and $c(3) \leq 9/5$, while improving the previously known estimates in every dimension $d \geq 4$.

In the plane, Falconer's distance conjecture predicts the much stronger conclusion $c(2) = 1$. Curiously, the works of Wolff, Erdoğan, and Du–Zhang (Wolff 1999; Erdogan 2005; Du and Zhang 2019), despite relying on substantially different methods, all arrived at the same bound

$$c(2) \leq \frac{4}{3}.$$

The repeated appearance of $4/3$ made it seem like a genuine barrier for the available techniques. It was therefore a major breakthrough when Guth, Iosevich, Ou, and Wang (Guth et al. 2020) broke this barrier in 2020.

Theorem 1 *Let $E \subset \mathbb{R}^2$ be compact. If the Hausdorff dimension of E is greater than $5/4$, then $\Delta(E)$ has positive Lebesgue measure.*

Equivalently, Theorem 1 proves that

$$c(2) \leq \frac{5}{4}.$$

Although this still lies $1/4$ above the conjectural threshold, it was the first improvement on Wolff's $4/3$ bound in more than two decades.

The argument also opened the way to progress in higher dimensions. In 2021, Du, Iosevich, Ou, Wang, and Zhang (Du, Iosevich, et al. 2021) extended the $1/4$ improvement to every even dimension $d \geq 4$, proving that

$$c(d) \leq \frac{d}{2} + \frac{1}{4}.$$

In 2023, Du, Ou, Ren, and Zhang (Du et al. 2023) improved this further, showing that for every $d \geq 3$,

$$c(d) \leq \frac{d^2 + d}{2d + 1} = \frac{d}{2} + \frac{1}{4} - \frac{1}{8d + 4}.$$

Thus, Wang's work on the planar problem not only broke a long-standing barrier in dimension two, but also helped initiate a new sequence of advances toward Falconer's conjectured threshold in higher dimensions.

References

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