The distortion of a knot can be arbitrarily large

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This is the second post in a series begun here, devoted to presenting one striking and comparatively accessible theorem associated with each of the 2026 Fields Medalists and the 2026 Abacus Medalist. This post focuses on John Pardon, one of the 2026 Fields Medalists, and on a question about knots posed by Gromov in 1983. Remarkably, Pardon found the solution while he was a senior undergraduate at Princeton; the resulting paper appeared in the *Annals of Mathematics* in 2011 (Pardon 2011).

We have discussed knots before in this blog, for instance here and here. Recall that a knot representative is a closed, non-self-intersecting curve

$$\gamma \subset \mathbb{R}^3.$$

More precisely, one may regard γ as the image of an embedding $S^1 \hookrightarrow \mathbb{R}^3$. Two representatives describe the same *knot type* if one can be continuously deformed into the other through embedded curves, or, equivalently, if they are ambient isotopic.

There are many ways to draw the same knot type, some much more contorted than others. The *distortion* measures how inefficiently a particular representative sits in space. If γ is rectifiable, its distortion is

$$\delta(\gamma) = \sup_{\substack{x,y\in\gamma \\ x\neq y}} \frac{d_\gamma(x,y)}{\|x-y\|},$$

where $d_\gamma(x,y)$ is the length of the shorter of the two arcs of γ joining x to y, while $\|x-y\|$ is their ordinary Euclidean distance in $\mathbb{R}^3$.

One can imagine an ant constrained to travel along the knot and a bird allowed to fly directly from x to y. The quotient above compares the ant's journey with the bird's shortcut. Thus large distortion means that some two points are close together in space but very far apart when one is forced to travel along the knot.

The distortion is unchanged if the entire curve is rescaled, since both the numerator and the denominator are multiplied by the same factor. It is always at least 1, although even the round circle has distortion $\pi/2$, attained by a pair of antipodal points.

The distortion of a knot type K is defined by taking the best possible representative:

$$\delta(K) = \inf_{\gamma \in K} \delta(\gamma),$$

where the infimum is over all rectifiable representatives of K. It is important to distinguish $\delta(\gamma)$ from $\delta(K)$. A particular drawing of a knot can have enormous distortion for accidental geometric reasons, whereas $\delta(K)$ asks how small the distortion can be after the knot has been rearranged in every possible way.

In 1983, Gromov asked whether every knot type in $\mathbb{R}^3$ has a representative of distortion less than 100 (Gromov et al. 1983). The particular number 100 is not the essential point: Gromov was asking whether there is any universal constant that bounds the distortion of all knot types.

To describe Pardon's answer, consider coprime positive integers p and q. The curve

$$\gamma_{p,q}(\phi) = \big((2+\cos(q\phi))\cos(p\phi),\ (2+\cos(q\phi))\sin(p\phi),\ -\sin(q\phi)\big), \qquad 0 \le \phi < \tag{1}$$

lies on the torus

$$\big(\sqrt{x^2+y^2} - 2\big)^2 + z^2 = 1.$$

As ϕ travels once around the circle, the curve winds p times around the central axis of the torus and q times around its tube. The assumption that p and q are coprime ensures that the result is a single closed curve rather than a link with several components. The knot type represented by (1) is called the (p,q)-*torus knot* and is denoted by $T_{p,q}$.

Pardon's theorem gives a quantitative lower bound for the distortion of these knots.

Theorem 1 *For coprime positive integers p and q,*

$$\delta(T_{p,q}) \ge \frac{1}{160} \min\{p,q\}.$$

Consequently, for every $C > 0$ there exists a knot type K such that

$$\delta(\gamma) > C$$

for every rectifiable representative γ of K.

Indeed, consecutive integers are coprime, so Theorem 1 gives

$$\delta(T_{n,n+1}) \ge \frac{n}{160} \longrightarrow \infty \qquad \text{as } n \longrightarrow \infty.$$

Given $C > 0$, it is enough to choose $n > 160C$. This answers Gromov's question not only for the proposed bound 100, but for every proposed universal bound.

The numerical constant $1/160$ comes from the quantitative estimates in this cutting argument. Its precise value is less important than the linear growth in $\min\{p, q\}$. Pardon's theorem shows that a quantity defined using only arclength and straight-line distance can retain enough topological information to distinguish knot types of arbitrarily large geometric complexity.

References

Gromov, Mikhael et al. 1983. "Filling Riemannian Manifolds." *J. Differential Geom.* 18 (1): 1–147.

Pardon, John. 2011. "On the Distortion of Knots on Embedded Surfaces." *Ann. of Math.* 174 (1): 637–46.