

Derivation of the Boltzmann equation

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The 2026 International Congress of Mathematicians is taking place in Philadelphia from 23–30 July. At the opening ceremony, the recipients of the Fields Medals and the Abacus Medal were announced. In this series of posts, I will present one striking and accessible theorem associated with each medalist. This first post focuses on Yu Deng, one of the 2026 Fields medalists, and on a landmark theorem that he proved jointly with Zaher Hani and Xiao Ma (Deng et al. 2025b).

In principle, Newton’s equations determine the trajectory of every particle in a gas. A real gas, however, contains an enormous number of particles, and following them individually is neither practical nor especially informative. Instead, one studies a statistical density

$$f(t, x, v),$$

where

$$f(t, x, v) \, dx \, dv$$

represents, approximately, the proportion of particles whose positions and velocities at time t lie near $x \in \mathbb{R}^d$ and $v \in \mathbb{R}^d$, respectively. The fundamental question is whether an evolution equation for this statistical density can be rigorously deduced from the Newtonian motion of the particles.

Let us begin with the microscopic model. Fix $d \geq 2$, and consider N identical hard spheres of diameter $\varepsilon > 0$, with centres $x_i \in \mathbb{R}^d$ and velocities $v_i \in \mathbb{R}^d$. We write

$$z_i = (x_i, v_i)$$

for the state of the i -th particle. Since the spheres cannot overlap, the state of the system belongs to

$$D_N^\varepsilon = \{ (z_1, \dots, z_N) \in (\mathbb{R}^{2d})^N \mid |x_i - x_j| \geq \varepsilon \text{ whenever } i \neq j \}.$$

Between collisions, the particles move freely:

$$\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = 0.$$

Suppose that particles i and j touch while approaching one another. Thus

$$|x_i - x_j| = \varepsilon, \quad (v_i^- - v_j^-) \cdot \omega < 0, \quad \omega = \frac{x_i - x_j}{\varepsilon} \in \mathbb{S}^{d-1},$$

where the superscript $-$ denotes the velocity immediately before the collision.

The velocities immediately after the collision are given by

$$\begin{aligned} v_i^+ &= v_i^- - ((v_i^- - v_j^-) \cdot \omega) \omega, \\ v_j^+ &= v_j^- + ((v_i^- - v_j^-) \cdot \omega) \omega. \end{aligned} \quad (1)$$

This collision rule conserves both total momentum and total kinetic energy. It is also time-reversible: if all velocities are reversed, the particles retrace their trajectories. Except for a set of initial configurations of Lebesgue measure zero, these rules define a unique motion for all time.

To obtain a statistical statement, we choose the initial configuration at random.

Let

$$f_0 : \mathbb{R}^{2d} \longrightarrow [0, +\infty)$$

satisfy

$$\int_{\mathbb{R}^{2d}} f_0(x, v) \, dx \, dv = 1.$$

The number of particles is itself random. For every $n \geq 0$ and every measurable set $A \subseteq D_n^\varepsilon$, assume that

$$\mathbb{P}_\varepsilon(N = n, (z_1, \dots, z_n) \in A) = \frac{1}{Z_\varepsilon} \frac{\varepsilon^{-(d-1)n}}{n!} \int_A \prod_{j=1}^n f_0(z_j) \, dz_1 \cdots dz_n, \quad (2)$$

where Z_ε is the normalizing constant obtained by summing over all $n \geq 0$. This probability distribution is called a *grand canonical ensemble*.

The factor $\varepsilon^{-(d-1)n}$ is chosen so that

$$\mathbb{E}_\varepsilon[N] \varepsilon^{d-1} \longrightarrow 1 \quad \text{as } \varepsilon \rightarrow 0. \quad (3)$$

This is the *Boltzmann–Grad scaling*. The expected number of particles is therefore of order

$$\varepsilon^{-(d-1)},$$

and tends to infinity as their diameter tends to zero. The total volume occupied by the particles is of order

$$\mathbb{E}_\varepsilon[N] \varepsilon^d \asymp \varepsilon,$$

and hence vanishes in the limit. On the other hand, their combined collision cross-section is of order

$$\mathbb{E}_\varepsilon[N] \varepsilon^{d-1} \asymp 1.$$

Thus the gas becomes increasingly dilute, but collisions remain visible on time scales of order one.

For every positive integer s , the s -particle correlation function

$$f_s^\varepsilon(t, z_1, \dots, z_s)$$

is defined by requiring that, for every bounded continuous function

$$\Phi : (\mathbb{R}^{2d})^s \longrightarrow \mathbb{R},$$

one has

$$\begin{aligned} & \int_{(\mathbb{R}^{2d})^s} \Phi(z_1, \dots, z_s) f_s^\varepsilon(t, z_1, \dots, z_s) \, dz_1 \cdots dz_s \\ &= \mathbb{E}_\varepsilon \left[\varepsilon^{(d-1)s} \sum_{\substack{i_1, \dots, i_s \\ \text{pairwise distinct}}} \Phi(z_{i_1}(t), \dots, z_{i_s}(t)) \right]. \end{aligned} \quad (4)$$

The sum is over ordered s -tuples of distinct particles. In particular, f_1^ε describes the statistical state of one typical particle.

The desired limiting behaviour is a factorization of the form

$$f_s^\varepsilon(t, z_1, \dots, z_s) \approx \mathbf{1}_{D_s^\varepsilon}(z_1, \dots, z_s) \prod_{j=1}^s f(t, z_j).$$

Apart from the unavoidable exclusion of overlapping configurations, this says that any fixed collection of particles is approximately independent and that each particle has the same one-particle density $f(t, \cdot)$. This property is called *propagation of chaos*. It is the rigorous counterpart of Boltzmann's molecular-chaos assumption.

Boltzmann proposed in 1872 that, under the scaling (3), the limiting one-particle density should satisfy the hard-sphere Boltzmann equation

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f(t, x, v) = Q(f, f)(t, x, v), \\ f(0, x, v) = f_0(x, v), \end{cases} \quad (5)$$

where

$$\begin{aligned} Q(f, f)(t, x, v) &= \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} ((v - v_*) \cdot \omega)_+ \\ &\quad \times \left[f(t, x, v') f(t, x, v'_*) - f(t, x, v) f(t, x, v_*) \right] \, d\omega \, dv_*. \end{aligned}$$

Here

$$a_+ = \max\{a, 0\},$$

and

$$v' = v - ((v - v_*) \cdot \omega)\omega, \quad v'_* = v_* + ((v - v_*) \cdot \omega)\omega.$$

The transport term $v \cdot \nabla_x f$ describes the free motion of the particles. The two terms in $Q(f, f)$ describe, respectively, the gain and loss of particles with velocity v caused by collisions.

The passage from the reversible system (1) to the irreversible equation (5) became one of the central problems of mathematical physics. It is closely connected with Hilbert's sixth problem, in which Hilbert asked for a rigorous passage from atomistic mechanics to the equations governing continuous media.

Grad's work in the 1950s identified the appropriate low-density scaling, and later work by Cercignani and others formulated the problem through the hierarchy of many-particle densities. In 1975, Lanford proved that hard-sphere dynamics converges to the Boltzmann equation, but only on a sufficiently short time interval (Lanford 1975). Many subsequent results refined Lanford's theorem or treated special regimes, including gases close to vacuum and gases at exact equilibrium. For general non-equilibrium initial data, however, the essential short-time restriction remained for almost fifty years.

We can now state the result of Deng, Hani, and Ma (Deng et al. 2025b). For $\beta > 0$ and a function $g = g(x, v)$, define

$$\|g\|_{\text{Bol}, \beta} = \sum_{k \in \mathbb{Z}^d} \sup_{\substack{|x-k| \leq 1 \\ v \in \mathbb{R}^d}} e^{\beta|v|^2} |g(x, v)|.$$

This norm requires Gaussian decay in the velocity variable, uniformly on unit-scale regions of physical space, and asks that the resulting local bounds be summable over all such regions.

Theorem 1 Fix $d \geq 2$ and $\beta > 0$. Let

$$f_0 : \mathbb{R}^{2d} \longrightarrow [0, +\infty)$$

satisfy

$$\int_{\mathbb{R}^{2d}} f_0(z) \, dz = 1$$

and

$$\|f_0\|_{\text{Bol}, 2\beta} + \|\nabla_x f_0\|_{\text{Bol}, 2\beta} < +\infty.$$

Suppose that the Boltzmann equation (5) has a non-negative solution f on a time interval $[0, T]$ satisfying

$$\sup_{0 \leq t \leq T} \sup_{x, v \in \mathbb{R}^d} e^{2\beta|v|^2} f(t, x, v) < +\infty.$$

Let the diameter- ε hard-sphere system be initialized according to the ensemble (2), and let f_s^ε be its correlation functions defined by (4). There exists a constant $\theta > 0$, depending only on d , such that, for all sufficiently small $\varepsilon > 0$, uniformly for $0 \leq t \leq T$ and all integers

$$1 \leq s \leq |\log \varepsilon|,$$

one has

$$\int_{(\mathbb{R}^{2d})^s} \left| f_s^\varepsilon(t, z_1, \dots, z_s) - \mathbf{1}_{D_s^\varepsilon}(z_1, \dots, z_s) \prod_{j=1}^s f(t, z_j) \right| \, dz_1 \cdots dz_s \leq \varepsilon^\theta.$$

The required upper bound on ε may depend on d , T , β , and the bounds appearing in the hypotheses.

When $s = 1$, the exclusion condition is empty, since

$$D_1^\varepsilon = \mathbb{R}^{2d}.$$

Theorem 1 therefore states directly that the one-particle density of the Newtonian gas converges in L^1 , uniformly on $[0, T]$, to the solution of the Boltzmann equation.

For larger values of s , the theorem says considerably more. Although collisions create correlations between particles at every fixed value of ε , these correlations become negligible in the Boltzmann–Grad limit. Molecular chaos is therefore propagated throughout the whole interval $[0, T]$.

Moreover, the theorem does not merely control each fixed number of particles. Since

$$|\log \varepsilon| \longrightarrow +\infty \quad \text{as } \varepsilon \rightarrow 0,$$

it controls collections whose size grows as the particles become smaller. The factor

$$\mathbf{1}_{D_s^\varepsilon}$$

records only the unavoidable fact that hard spheres cannot overlap. No perturbative smallness assumption is imposed on f_0 ; the conclusion is conditional only on the stated regularity and on the existence of a sufficiently well-behaved solution of the Boltzmann equation.

The phrase “arbitrarily long time” must be interpreted carefully. The theorem does not give one estimate that is uniform for every $t \geq 0$. Instead, T may be any fixed time up to which the Boltzmann solution exists and satisfies the stated bound. The required smallness of ε is allowed to depend on T . Consequently, if the Boltzmann solution remains regular for all time, the derivation is valid on every finite time interval. The theorem does not, by itself, settle the separate and difficult problem of global regularity for the Boltzmann equation with completely general initial data.

This result removes the long-standing short-time barrier in the passage from Newton’s laws to Boltzmann’s kinetic equation. Combined with the authors’ companion work (Deng et al. 2025a), which connects the Boltzmann equation to the equations of fluid mechanics, it realizes the Newton–Boltzmann–fluid program associated with Hilbert’s sixth problem in the rarefied hard-sphere setting considered by the authors.

The theorem also gives a precise setting in which irreversible statistical behaviour emerges from reversible microscopic mechanics. There is no contradiction with the reversibility of the hard-sphere system. The theorem begins with an asymptotically uncorrelated random ensemble; the time reversal of a typical evolved state contains highly organized correlations and does not belong to the same class of initial ensembles. The preferred direction of time therefore enters through the statistical initial conditions and the limiting procedure, rather than through any failure of Newton’s laws themselves.

In 2026, Deng received a Fields Medal for his work in partial differential equations. The official citation explicitly highlighted the rigorous derivation of the Boltzmann equation from hard-sphere dynamics (International Mathematical Union 2026).

References

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