The Jacobian conjecture is false

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On July 19, 2026, mathematician Levent Alpöge announced an explicit counterexample to the Jacobian conjecture. Formulated by O.-H. Keller in 1939, the conjecture had remained open for eighty-seven years and appears as Problem 16 on Stephen Smale's influential list *Mathematical Problems for the Next Century*. Alpöge credited Akhil Mathew with raising the question and Anthropic's AI system Claude Fable 5 with the work that led to the example (Alpöge 2026). The result is especially striking because the counterexample is explicit and its decisive properties can be verified by direct calculation.

The polynomial ring

$$\mathbb{C}[x_1, \ldots, x_n]$$

consists of all polynomials in the variables $x_1, \ldots, x_n$ with complex coefficients. A $\mathbb{C}$-algebra endomorphism of this ring is a map

$$\varphi : \mathbb{C}[x_1, \ldots, x_n] \longrightarrow \mathbb{C}[x_1, \ldots, x_n]$$

that preserves addition, multiplication, and the element 1, and satisfies

$$\varphi(c) = c \qquad \text{for every } c \in \mathbb{C}.$$

Such a map is completely determined by the images of the variables,

$$f_i = \varphi(x_i), \qquad 1 \le i \le n.$$

Conversely, any choice of polynomials $f_1, \ldots, f_n$ defines an endomorphism by substitution:

$$\varphi_F(h) = h(f_1, \ldots, f_n).$$

The same collection of polynomials also defines a polynomial map

$$F = (f_1, \ldots, f_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^n.$$

The endomorphism φ_F is an automorphism precisely when the associated polynomial map F has a polynomial inverse. More explicitly, this means that there is a polynomial map

$$G : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

such that

$$G \circ F = F \circ G = \mathrm{id}_{\mathbb{C}^n} .$$

Indeed, if

$$\varphi_F^{-1}(x_i) = g_i,$$

then $G = (g_1, \ldots, g_n)$ is a polynomial inverse of F. Conversely, any polynomial inverse of F defines the inverse endomorphism of φ_F.

For a polynomial map $F = (f_1, \ldots, f_n)$, its Jacobian matrix is

$$J_F = \left(\frac{\partial f_i}{\partial x_j} \right)_{1 \leq i,j \leq n},$$

and its Jacobian determinant is $\det J_F$. If F has a polynomial inverse G, then the chain rule gives

$$J_G(F(x)) J_F(x) = I_n,$$

where I_n is the $n \times n$ identity matrix. Taking determinants yields

$$(\det J_G)(F(x)) \det J_F(x) = 1.$$

The only units in $\mathbb{C}[x_1, \ldots, x_n]$ are the nonzero constants. Consequently, a nonzero constant Jacobian determinant is a necessary condition for F to have a polynomial inverse.

The Jacobian conjecture, originating in Keller's 1939 paper (Keller 1939), asserted that this necessary condition was also sufficient: every polynomial map

$$F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

with nonzero constant Jacobian determinant should have a polynomial inverse. A map satisfying the Jacobian condition is often called a *Keller map*; see (Bass et al. 1982) for the classical theory.

The conjecture was therefore a local-to-global assertion. A nonvanishing Jacobian determinant makes F locally invertible at every point, but the conjecture claimed that this local condition forces a single global polynomial inverse. The following example shows that, beginning in dimension three, it does not.

Theorem 1 *Let*

$$F = (P, Q, R) : \mathbb{C}^3 \longrightarrow \mathbb{C}^3,$$

where

$$\begin{aligned} P(x, y, z) &= (1 + xy)^3 z + y^2(1 + xy)(4 + 3xy), \\ Q(x, y, z) &= y + 3x(1 + xy)^2 z + 3xy^2(4 + 3xy), \\ R(x, y, z) &= 2x - 3x^2 y - x^3 z. \end{aligned}$$

Then

$$\det J_F = -2,$$

but the three distinct points

$$\left(0, 0, -\frac{1}{4}\right), \qquad \left(1, -\frac{3}{2}, \frac{13}{2}\right), \qquad \left(-1, \frac{3}{2}, \frac{13}{2}\right)$$

have the same image:

$$F\left(0, 0, -\frac{1}{4}\right) = F\left(1, -\frac{3}{2}, \frac{13}{2}\right) = F\left(-1, \frac{3}{2}, \frac{13}{2}\right) = \left(-\frac{1}{4}, 0, 0\right).$$

Consequently, F is not injective and therefore has no polynomial inverse. Equivalently, the endomorphism of $\mathbb{C}[x, y, z]$ defined by

$$x \longmapsto P, \qquad y \longmapsto Q, \qquad z \longmapsto R$$

is not an automorphism.

Proof. Set

$$A = 1 + xy, \qquad B = A^2 z + y^2(4 + 3xy).$$

Then

$$P = AB, \qquad Q = y + 3xB.$$

On the dense open set $A \neq 0$, define

$$s = \frac{x}{A}.$$

A direct calculation gives

$$Q = y + 3Ps, \qquad R = 2s - ys^2 - Ps^3.$$

The two corresponding changes of coordinates have Jacobian determinants

$$\det\left(\frac{\partial(P, y, s)}{\partial(x, y, z)}\right) = -P_z s_x = -A$$

and

$$\det\left(\frac{\partial(P, Q, R)}{\partial(P, y, s)}\right) = 2(1 - ys) = \frac{2}{A}.$$

It follows that

$$\det J_F = \det\left(\frac{\partial(P, Q, R)}{\partial(P, y, s)}\right) \det\left(\frac{\partial(P, y, s)}{\partial(x, y, z)}\right) = \frac{2}{A}(-A) = -2$$

whenever $A \neq 0$. Since $\det J_F$ is a polynomial, the identity extends to all of $\mathbb{C}^3$.

At $(0, 0, -1/4)$ and $(1, -3/2, 13/2)$, the pairs (A, B) are respectively

$$\left(1, -\frac{1}{4}\right) \qquad \text{and} \qquad \left(-\frac{1}{2}, \frac{1}{2}\right),$$

and direct substitution gives

$$(P, Q, R) = \left(-\frac{1}{4}, 0, 0\right)$$

in both cases. Finally, the involution

$$(x, y, z) \longmapsto (-x, -y, z)$$

fixes P and changes the signs of Q and R. It sends

$$\left(1, -\frac{3}{2}, \frac{13}{2}\right) \quad \text{to} \quad \left(-1, \frac{3}{2}, \frac{13}{2}\right),$$

so the third point has the same image as well. □

All coefficients of the map and all three witness points are rational, so the same counterexample works over every field of characteristic zero.

For every $n > 3$, adjoining unchanged coordinates produces a counterexample in dimension n:

$$(x, y, z, u_4, \ldots, u_n) \longmapsto \big(P(x, y, z), Q(x, y, z), R(x, y, z), u_4, \ldots, u_n\big).$$

Its Jacobian matrix is block diagonal with blocks J_F and I_{n-3}, and hence its Jacobian determinant is still -2. The map is also still noninjective. Therefore the Jacobian conjecture is false in every dimension $n \geq 3$. The one-dimensional case is elementary, while the two-dimensional Jacobian conjecture remains open.

Alpöge, Levent. 2026. *Hello There the Jacobian Conjecture Is False*. https://x.com/__alpoge__/status/2079028340955197566.

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