

Arbitrarily long arithmetic progressions in primes

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This post continues my series on favorite theorems of the twenty-first century. For an overview of the categories and my earlier selections, see [this post](#).

My choice for 2004 in number theory—and, indeed, across all areas of mathematics—is the theorem of Green and Tao that the primes contain arbitrarily long arithmetic progressions. Green and Tao announced their proof in 2004, and the final paper appeared in the *Annals of Mathematics* in 2008 (Green and Tao 2008).

In a [previous blog post](#), we discussed the following fundamental question. Given polynomials

$$P_i(x_1, \dots, x_n), \quad i = 1, \dots, k,$$

in n variables with integer coefficients, under what conditions are there infinitely many integer tuples $(x_1, \dots, x_n)$ for which all the values

$$P_1(x_1, \dots, x_n), \dots, P_k(x_1, \dots, x_n)$$

are simultaneously prime?

In 1837, Dirichlet proved that if a and b are coprime integers, then the linear polynomial $ax + b$ takes prime values for infinitely many integers x . This settles the question for a single linear polynomial in one variable. What happens when one considers several linear polynomials simultaneously?

A particularly natural special case is given by

$$P_i(x, y) = x + (i - 1)y, \quad i = 1, \dots, k,$$

together with the condition $y \neq 0$. Requiring all these polynomials to be prime is precisely the same as asking for a nonconstant k -term arithmetic progression of primes:

$$x, x + y, x + 2y, \dots, x + (k - 1)y.$$

In 1939, van der Corput (Corput 1939) proved that the set $\mathcal{P}$ of primes contains infinitely many three-term arithmetic progressions. In 1953, Roth (Roth 1953) proved that every subset of the integers with positive upper density contains a

three-term arithmetic progression. More than fifty years later, Green (Green 2005) established a common generalization of these two results: every subset of the primes with positive relative upper density contains a nonconstant three-term arithmetic progression.

Here a subset $A \subseteq \mathcal{P}$ has positive relative upper density in the primes if

$$\limsup_{N \rightarrow \infty} \frac{|A \cap \mathcal{P}_N|}{|\mathcal{P}_N|} > 0,$$

where $\mathcal{P}_N$ denotes the set of primes less than N .

In 1975, Szemerédi (Szemerédi 1975) generalized Roth’s theorem to progressions of arbitrary finite length. He proved that every subset of the integers with positive upper density contains k -term arithmetic progressions for every positive integer k . This result, however, cannot be applied directly to the primes, since the primes have density zero among the integers.

For almost seventy years after van der Corput’s theorem, no one was able to prove that the primes contain even a four-term arithmetic progression. Green and Tao overcame this obstacle and established the result for progressions of every finite length (Green and Tao 2008).

Theorem 1 (Green–Tao) *The primes contain arbitrarily long arithmetic progressions. More generally, every subset of the primes with positive relative upper density contains arithmetic progressions of every finite length.*

The main difficulty is that Szemerédi’s theorem applies to dense subsets of the integers, whereas the primes become increasingly sparse. Green and Tao bridged this gap by proving a version of Szemerédi’s theorem inside suitably pseudorandom sets. This result is now known as the *relative Szemerédi theorem*.

Informally, the relative Szemerédi theorem says that if S is a sufficiently pseudorandom set of integers and $A \subseteq S$ has positive relative density in S , then A contains arbitrarily long arithmetic progressions. Green and Tao then constructed a pseudorandom majorant for the primes: a sufficiently uniform ambient object within which the primes form a subset of positive relative density. Applying the relative Szemerédi theorem in this setting yields arbitrarily long arithmetic progressions of primes.

References

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