

Stable Bernstein in Dimensions 4,5,6

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The July 2026 issue of the *Annals of Mathematics* contains a paper by Chodosh, Li, Minter, and Stryker (Chodosh et al. 2026) that resolves the stable Bernstein problem in $\mathbb{R}^5$. To place this result in context, we first recall the relevant notions and the history of the problem.

A k -dimensional manifold $M \subset \mathbb{R}^d$ is called a *minimal manifold* if every point $p \in M$ has a neighbourhood $S \subset M$, with boundary $B = \partial S$, such that the k -dimensional volume of S is no greater than that of any other k -dimensional manifold S' with boundary B . When $k = d - 1$, the manifold M is called a *minimal hypersurface*.

This definition is local: it does not imply that every relatively compact domain $S \subset M$ minimizes volume among all manifolds with the same boundary. A manifold satisfying this stronger property is called *volume-minimizing*; in dimension two, it is called an *area-minimizing surface*. The classical Plateau problem, first posed in mathematical form by Lagrange in 1760, asks whether every simple closed curve $\Gamma \subset \mathbb{R}^3$ spans a surface of least area. In 1931, Douglas (Douglas 1931) solved the classical disk-type Plateau problem. In 1961, De Giorgi (De Giorgi 1961) established a far-reaching higher-dimensional analogue: for $3 \leq d \leq 7$, every smooth, closed, oriented $(d - 2)$ -dimensional submanifold $\Gamma \subset \mathbb{R}^d$ bounds a smooth, compact, volume-minimizing hypersurface in $\mathbb{R}^d$. In dimensions $d \geq 8$, such a minimizer need not be smooth for every choice of Γ . Nevertheless, at least when $d \leq 11$, smoothness continues to hold for “generic” boundary data; see (Chodosh et al. 2025).

Every volume-minimizing manifold is minimal, but the converse is false. For minimal graphs, however, the distinction disappears. An *entire minimal graph* in $\mathbb{R}^d$ is a hypersurface of the form

$$M = \{(x, u(x)) : x \in \mathbb{R}^{d-1}\}$$

with vanishing mean curvature. Every such graph is volume-minimizing.

An important necessary condition for volume minimization is stability. A minimal manifold M is called *stable* if, for every relatively compact domain $S \subset M$ and every normal variation of S that fixes ∂S , the second variation of the volume functional is non-negative. Every volume-minimizing manifold is stable, although the converse again fails. In particular, every minimal graph is stable.

In 1916, Bernstein proved that every entire minimal graph in $\mathbb{R}^3$ is a plane. The celebrated Bernstein problem asks whether the same conclusion holds in higher dimensions: must every entire minimal graph in $\mathbb{R}^d$ be a hyperplane? The answer is affirmative for $d \leq 8$ and negative for $d \geq 9$; see (Simons 1968; Bombieri et al. 1969).

The *stable Bernstein problem* asks for which dimensions $d \leq 8$ every complete, stable minimal hypersurface in $\mathbb{R}^d$ must be a hyperplane. Since stability is substantially weaker than the assumption of being a graph, this is a far-reaching extension of the original Bernstein problem. In 1969, Bombieri, De Giorgi, and Giusti (Bombieri et al. 1969) produced a counterexample in $\mathbb{R}^8$. By contrast, do Carmo and Peng (Carmo and Peng 1979) proved in 1979 that the answer is affirmative in $\mathbb{R}^3$. Despite considerable attention and numerous partial results, the cases $4 \leq d \leq 7$ remained open for several decades.

In 2024, Chodosh and Li (Chodosh and Li 2024) settled the first of these cases by proving the following theorem.

Theorem 1 *Let M be an immersed, complete, connected, two-sided, stable minimal hypersurface in $\mathbb{R}^4$. Then M is a Euclidean hyperplane.*

The proof of Theorem 1 initiated a period of rapid progress. Two alternative proofs soon appeared in (Chodosh and Li 2023) and (Catino et al. 2024). Each exploited the assumption $d = 4$ in a different way, but the argument in (Chodosh and Li 2023) appeared particularly well suited to generalization. Chodosh, Li, Minter, and Stryker (Chodosh et al. 2026) succeeded in extending that argument to $\mathbb{R}^5$, thereby proving the following result.

Theorem 2 *Let M be an immersed, complete, connected, two-sided, stable minimal hypersurface in $\mathbb{R}^5$. Then M is a Euclidean hyperplane.*

A further extension by Mazet (Mazet 2024) established the corresponding result in $\mathbb{R}^6$. Thus, among the dimensions not already ruled out by known counterexamples, only the case of $\mathbb{R}^7$ remains open.

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