Counting flats of a matroid

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The work of Braden, Huh, Matherne, Proudfoot, and Wang (Braden et al. 2020), which grew out of their 2020 preprint and has recently been accepted to the *Journal of the American Mathematical Society*, proves a far-reaching form of a classical incidence phenomenon.

Recall that a hyperplane in $\mathbb{R}^d$ is a $(d-1)$-dimensional affine subspace. More generally, if $Q_1, \ldots, Q_m \in \mathbb{R}^d$, their affine span is the smallest affine subspace of $\mathbb{R}^d$ containing them; a finite point set P is said to determine an affine subspace S if S is the affine span of some subset of P.

A classical theorem in plane geometry says that n points in the plane, not all lying on a single line, determine at least n lines. In 1951, Motzkin (Motzkin 1951) proved a higher-dimensional version of the plane theorem: n points in $\mathbb{R}^d$, not all lying on a single hyperplane, determine at least n hyperplanes. In 1970, Greene (Greene 1970) strengthened this result by proving that one can choose these hyperplanes in a pointwise compatible way: if $P = \{P_1, \ldots, P_n\}$ and H is the set of hyperplanes determined by P, then there is an injective map $\psi : P \to H$ such that $P_i \in \psi(P_i)$ for every $i = 1, \ldots, n$. In 2017, Huh and Wang (Huh and Wang 2017) generalized Greene's theorem from points and hyperplanes to subspaces of arbitrary dimensions.

Theorem 1 *Let $P = \{P_1, \ldots, P_n\}$ be a finite set of points that affinely spans $\mathbb{R}^d$, and let F_r be the set of $(r-1)$-dimensional affine subspaces of $\mathbb{R}^d$ determined by P. If*

$$1 \leq k \leq j \leq d + 1 - k,$$

then there is an injective map

$$\psi : F_k \longrightarrow F_j$$

such that $S \subseteq \psi(S)$ for every $S \in F_k$. In particular,

$$|F_k| \leq |F_j|.$$

The case $k = 1$ and $j = d$ recovers Greene's theorem: the elements of F_1 are the points of P, while the elements of F_d are the hyperplanes determined by P. Braden, Huh, Matherne, Proudfoot, and Wang (Braden et al. 2020) proved a much broader version of this phenomenon for arbitrary matroids.

Let $M = (X, I)$ be a matroid on a finite ground set X. For a subset $A \subseteq X$, its rank is

$$r(A) = \max\{|B| : B \in I,\ B \subseteq A\},$$

and the rank of the matroid is $r(M) = r(X)$. A subset $S \subseteq X$ is a flat of rank k if $r(S) = k$ and every element outside S increases the rank by one, that is,

$$r(S \cup \{x\}) = r(S) + 1 \qquad \text{for every } x \in X \setminus S.$$

Let $F_k(M)$ denote the set of flats of rank k in M.

Theorem 2 *Let M be a matroid. If*

$$0 \leq k \leq j \leq r(M) - k,$$

then there is an injective map

$$\psi : F_k(M) \longrightarrow F_j(M)$$

such that $S \subseteq \psi(S)$ for every $S \in F_k(M)$. In particular,

$$|F_k(M)| \leq |F_j(M)|.$$

Taking $j = r(M) - k$ in Theorem 2 gives

$$|F_k(M)| \leq |F_{r(M)-k}(M)|$$

whenever $k \leq r(M)/2$. This inequality was conjectured by Dowling and Wilson in 1974 and became known as the Dowling–Wilson conjecture, or the top-heavy conjecture. The Huh–Wang theorem proves the conjecture for matroids realizable over a field, while Theorem 2 proves it for all matroids.

The proof of Theorem 2 follows the Hodge-theoretic strategy. The guiding analogy is with the cohomology of algebraic varieties: to a matroid one attaches a carefully constructed combinatorial substitute for a cohomology space, and then proves analogues of Poincaré duality, the hard Lefschetz theorem, and the Hodge–Riemann relations. In (Braden et al. 2020), the relevant object is the intersection cohomology module of a matroid. Once these Hodge-theoretic statements are established, the injective maps in Theorem 2 and the top-heavy inequalities follow.

References

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