Counting Closed Reeb Orbits

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The 2024 preprint of Çineli, Ginzburg, and Gürel (Cineli et al. 2024), now accepted by the *Annals of Mathematics*, settles a beautifully simple counting problem in Hamiltonian dynamics: on a smooth convex energy surface, how many genuinely different periodic motions must there be?

We begin by explaining the motion itself. Let n be a positive integer. The phase space for a system with n position variables and n momentum variables is the real vector space $\mathbb{R}^{2n}$. We write a point $x \in \mathbb{R}^{2n}$ as

$$x = (q_1, p_1, \ldots, q_n, p_n),$$

where q_j is the j-th position coordinate and p_j is the corresponding momentum coordinate.

Let $W \subset \mathbb{R}^{2n}$ be a compact convex set with smooth boundary, and write

$$M = \partial W$$

for that boundary. Here compact means closed and bounded, convex means that the line segment between any two points of W lies entirely in W, and smooth means that M has a well-defined tangent hyperplane at every point and these tangent hyperplanes vary smoothly. We also assume that the origin 0 lies in the interior of W. Under these hypotheses M is star-shaped with respect to the origin: for every $x \in W$, the segment from 0 to x lies in W.

For a point $x \in M$, the tangent space T_xM is the set of all velocity vectors of smooth curves on M passing through x. In other words, a vector v belongs to T_xM if there is a smooth curve $\eta : (-\varepsilon, \varepsilon) \to M$, for some $\varepsilon > 0$, such that

$$\eta(0) = x \qquad \text{and} \qquad \eta'(0) = v.$$

We now introduce two standard measuring devices on $\mathbb{R}^{2n}$. The first is the one-form λ_0. A one-form is a rule which, at each point x, takes a tangent vector v based at x and returns a real number. If

$$x = (q_1, p_1, \ldots, q_n, p_n)$$

and

$$v = (v_{q_1}, v_{p_1}, \ldots, v_{q_n}, v_{p_n}),$$

then

$$\lambda_{0,x}(v) = \frac{1}{2}\sum_{j=1}^{n}\left(q_j v_{p_j} - p_j v_{q_j}\right).$$

Geometrically, this measures how much the vector v points in the angular direction around the origin in the coordinate planes (q_j, p_j). For $n = 1$, the expression $q_1 v_{p_1} - p_1 v_{q_1}$ is the signed area of the parallelogram spanned by x and v; for larger n, we add this contribution over all coordinate planes.

The second measuring device is the two-form ω_0. A two-form is a rule which takes two vectors and returns a real number; it is bilinear, meaning linear in each input separately. For

$$u = (u_{q_1}, u_{p_1}, \ldots, u_{q_n}, u_{p_n})$$

and

$$v = (v_{q_1}, v_{p_1}, \ldots, v_{q_n}, v_{p_n}),$$

define

$$\omega_0(u, v) = \sum_{j=1}^{n}\left(u_{q_j} v_{p_j} - u_{p_j} v_{q_j}\right).$$

In each coordinate plane (q_j, p_j), the term $u_{q_j} v_{p_j} - u_{p_j} v_{q_j}$ is the signed area of the parallelogram spanned by the two projected vectors. The form ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$, the basic geometric structure behind Hamiltonian mechanics.

The Reeb vector field is the rule that tells us how to move along the surface M. At each point $x \in M$, the Reeb vector $R(x)$ is the unique tangent vector $R(x) \in T_x M$ satisfying

$$\lambda_{0,x}(R(x)) = 1$$

and

$$\omega_0(R(x), v) = 0 \qquad \text{for every } v \in T_x M.$$

The second condition chooses the special direction of motion on M. The first condition fixes the speed along that direction. Thus the Reeb vector field assigns one preferred velocity vector to every point of M.

The corresponding motion is called the Reeb flow. We denote it by

$$\varphi^t : M \to M,$$

where the superscript t records time. Starting from $x \in M$, the point $\varphi^t(x)$ is obtained by solving the differential equation

$$\frac{d}{dt}\varphi^t(x) = R(\varphi^t(x)), \qquad \varphi^0(x) = x.$$

So $\varphi^t(x)$ is the point reached after moving for time t with velocity prescribed by the Reeb vector field.

A closed Reeb orbit is a smooth map

$$\gamma : \mathbb{R} \to M$$

such that

$$\gamma'(t) = R(\gamma(t)) \qquad \text{for all } t \in \mathbb{R},$$

and such that there is a number $T > 0$ with

$$\gamma(t+T) = \gamma(t) \qquad \text{for all } t \in \mathbb{R}.$$

The number T is called a period of the orbit. Two closed Reeb orbits are regarded as the same if they trace the same geometric trajectory but with different starting times; that is, if one is obtained from the other by replacing t with $t + a$ for some constant a.

A closed Reeb orbit is called prime if it is not obtained by going several times around a shorter closed orbit. Thus running twice, three times, or k times around the same closed trajectory does not create a new prime orbit. Prime orbits are the basic unrepeated periodic motions of the Reeb flow.

In this setting, Rabinowitz (Rabinowitz 1978) and Weinstein (Weinstein 1978) independently proved in 1978 that at least one prime closed Reeb orbit must exist. The natural next question is not existence but counting. The multiplicity conjecture predicts that the Reeb flow on the boundary of a smooth compact convex domain in $\mathbb{R}^{2n}$ has at least n distinct prime closed Reeb orbits.

The number n cannot be improved in general. To see why, choose positive numbers $a_1, \dots, a_n$ such that

$$\frac{a_i}{a_j} \notin \mathbb{Q} \qquad \text{whenever } i \neq j,$$

and consider the ellipsoid

$$E(a_1, \dots, a_n) = \left\{ (q_1, p_1, \dots, q_n, p_n) \in \mathbb{R}^{2n} \;\middle|\; \sum_{j=1}^{n} \frac{q_j^2 + p_j^2}{a_j} \leq 1 \right\}.$$

On the boundary of this ellipsoid, the Reeb flow rotates in the coordinate planes (q_j, p_j). The irrationality condition above prevents motions in two different coordinate planes from closing up at the same time. As a result, the only prime closed Reeb orbits are the n coordinate circles

$$q_j^2 + p_j^2 = a_j, \qquad q_i = p_i = 0 \text{ for } i \neq j.$$

Thus some examples have exactly n prime closed Reeb orbits, so the lower bound predicted by the multiplicity conjecture is the best possible one.

Long and Zhu (Long and Zhu 2002) proved the multiplicity conjecture for convex W under an additional assumption called non-degeneracy. To explain this condition, take a closed orbit and choose a small hypersurface that cuts the orbit transversely at one point. Nearby points flow around and eventually return to this hypersurface, giving a first-return map. The orbit is non-degenerate if the derivative of this first-return map has no eigenvalue equal to 1. The Reeb flow is non-degenerate if all of its closed orbits are non-degenerate. Intuitively, non-degeneracy rules out closed orbits that sit inside smooth families of nearby closed orbits.

The breakthrough of Çineli, Ginzburg, and Gürel (Cineli et al. 2024) is that this extra non-degeneracy assumption is not needed in the convex case.

Theorem 1 *Let n be a positive integer. Let $W \subset \mathbb{R}^{2n}$ be a compact convex set with smooth boundary $M = \partial W$, and assume that 0 lies in the interior of W. Define the Reeb vector field R on M by the conditions*

$$\lambda_{0,x}(R(x)) = 1$$

and

$$\omega_0(R(x), v) = 0 \qquad \text{for every } v \in T_xM.$$

Then the Reeb flow of R on M has at least n distinct prime closed Reeb orbits.

Gürel (Gürel 2015) formulated an even sharper prediction, often called the n-or-∞ conjecture. It says that, for Reeb flows of the kind considered here, the number of distinct prime closed Reeb orbits should be either exactly n or infinite. In other words, the ellipsoid example above should be the only finite pattern: once there are more than the minimum number of prime orbits, there should be infinitely many.

The theorem of Çineli, Ginzburg, and Gürel proves the lower-bound half of this picture for convex domains. In the same paper, they also confirm the n-or-∞ conjecture under two additional assumptions: the domain is centrally symmetric, meaning

$$x \in W \iff -x \in W,$$

and the Reeb flow is non-degenerate. Under those hypotheses, the flow has either exactly n distinct prime closed Reeb orbits or infinitely many.

References

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