Exotic projective planes in 4-sphere

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Miyazawa's 2023 preprint (Miyazawa 2023), now accepted by the *Annals of Mathematics*, constructs embeddings of the real projective plane in the 4-sphere that are topologically isotopic but not smoothly isotopic. In fact, it gives infinitely many such smooth embeddings, all equivalent from the topological point of view and all distinct from the smooth point of view.

Knot theory has appeared several times on this blog; see here and here. In those posts, we discussed ordinary knots: embedded copies of S^1 in $\mathbb{R}^3$. In dimension 4, one can also knot entire surfaces. A *surface knot* in S^4 is a smooth embedding

$$F : \Sigma \hookrightarrow S^4$$

of a closed surface Σ. As usual, we often do not distinguish between the embedding F and its image $F(\Sigma) \subset S^4$.

The simplest closed non-orientable surface is the real projective plane $\mathbb{RP}^2$, the space of all lines through the origin in $\mathbb{R}^3$. A P^2*-knot* is a smooth embedding

$$P : \mathbb{RP}^2 \hookrightarrow S^4,$$

where P stands for "projective". Two P^2-knots $P_0, P_1 : \mathbb{RP}^2 \hookrightarrow S^4$ are called *smoothly isotopic* if there is a continuous family of diffeomorphisms

$$H_t : S^4 \to S^4, \qquad 0 \le t \le 1,$$

such that H_0 is the identity and

$$H_1\big(P_0(\mathbb{RP}^2)\big) = P_1(\mathbb{RP}^2).$$

They are called *topologically isotopic* if the same condition holds with homeomorphisms in place of diffeomorphisms. In the topological category, the embedded surface is assumed to be *locally flat*: near every point, it looks topologically like the standard inclusion of $\mathbb{R}^2$ in $\mathbb{R}^4$. An *exotic pair of embeddings* is a pair that is topologically isotopic but not smoothly isotopic.

Before Miyazawa's work, exotic pairs of non-orientable surface knots in S^4 were known, but only for surfaces of larger non-orientable genus. For example, Finashin, Kreck, and Viro (Finashin et al. 1988) constructed exotic embeddings of the connected sum of 10 copies of $\mathbb{RP}^2$, and Finashin (Finashin 2009) later gave examples for the connected sum of 6 copies. By contrast, the case of a single projective plane remained mysterious. Miyazawa (Miyazawa 2023) proved that even this smallest case has infinitely many smooth forms.

Theorem 1 *There exists an infinite family*

$$P_0, P_1, P_2, \ldots$$

of smooth embeddings of $\mathbb{RP}^2$ *into* S^4 *such that all the* P_n *are mutually topologically isotopic, but no two distinct members of the family are smoothly isotopic.*

The theorem says that the embeddings P_n are all the same from the purely topological point of view: a homeomorphism of S^4, continuously deformable to the identity, can move any one of them to any other. Smoothly, however, they are all different. Moreover, no diffeomorphism of S^4, even one not assumed to be isotopic to the identity, can carry P_m to P_n when $m \neq n$. Thus the gap between topology and smooth topology already appears for the smallest non-orientable surface that embeds in S^4.

References

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