On Birkhoff's rigidity conjecture

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Berger's 2024 preprint (Berger 2024), now accepted by *Acta Mathematica*, gives a striking strengthening of the failure of Birkhoff's rigidity conjecture. It constructs analytic area-preserving maps of the sphere and the disk whose periodic points are as scarce as those of an irrational rotation, but whose global dynamics are far from rotation-like: each example has a dense orbit.

Birkhoff's rigidity conjecture was a long-standing conjecture in low-dimensional analytic dynamics. Its guiding question was whether an analytic area-preserving system with as few periodic orbits as a rigid rotation must itself be a rotation in disguise. To formulate this precisely, we recall the relevant definitions.

A *surface* is a two-dimensional space that locally looks like an open part of the plane; for the disk, boundary points locally look like points on the edge of a half-plane. An *oriented surface* is a surface on which a consistent positive direction has been chosen, analogous to declaring counterclockwise to be the positive direction in the plane. In this discussion we use the following standard oriented surfaces:

$$\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}, \qquad \mathbb{A} = \mathbb{T}\times\mathbb{R}, \qquad \mathbb{D} = \{(u,v) \in \mathbb{R}^2 \mid$$

Here $\mathbb{S}^2$ is the 2-sphere, $\mathbb{A}$ is the cylinder, $\mathbb{D}$ is the closed disk, and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle. A point of $\mathbb{T}$ is a real number considered modulo 1, so adding $\alpha \in \mathbb{T}$ means rotating a circle by angle $2\pi\alpha$.

A *diffeomorphism* $f : X \to X$ is a bijective map such that both f and f^{-1} are smooth, meaning differentiable to all orders in local coordinates; for the disk, smoothness is understood up to the boundary. The map is *orientation preserving* if it does not reverse the chosen orientation. It is *area preserving* if it preserves the standard area form on the surface: $d\theta \wedge dt$ on $\mathbb{A}$, $du \wedge dv$ on $\mathbb{D}$, and the usual spherical area form on $\mathbb{S}^2$. In this two-dimensional setting, a *symplectomorphism* is an orientation- and area-preserving diffeomorphism.

A map $f : X \to X$ is *analytic*, or *real-analytic*, if in local coordinates its coordinate functions are represented by convergent power series. At boundary points of $\mathbb{D}$, this is understood in boundary charts, equivalently in local half-plane coordinates. Thus an *analytic symplectomorphism* is a symplectomorphism whose local coordinate expressions are real-analytic.

Every self-map $f : X \to X$ defines a discrete-time dynamical system by iteration:

$$x_n = f^n(x_0), \qquad n = 0, 1, 2, \dots,$$

where $f^0 = \mathrm{id}$ and $f^{n+1} = f \circ f^n$. The *forward orbit* of a point $x \in X$ is

$$\mathrm{Orb}_f^+(x) = \{x, f(x), f^2(x), f^3(x), \dots\}.$$

A subset $Y \subset X$ is called *dense* in X if every nonempty open set in X contains a point of Y; equivalently, $\overline{Y} = X$, where the overline denotes closure. The map f is called *transitive* if at least one point has a dense forward orbit, that is, if

$$\overline{\mathrm{Orb}_f^+(x)} = X$$

for some $x \in X$.

A point $x \in X$ is called a *periodic point* if $f^n(x) = x$ for some positive integer n. The least such n, when it exists, is the period of x. Periodic points of period 1 are called *fixed points*. A *homeomorphism* is a continuous bijection with continuous inverse. Two dynamical systems $f : X \to X$ and $g : X \to X$ are called *topologically conjugate* if there exists a homeomorphism $h : X \to X$ such that

$$h \circ f = g \circ h,$$

where $\circ$ denotes composition. This means that the two systems are the same from a topological point of view: the homeomorphism h sends orbits of f to orbits of g.

The model examples are rigid rotations. For $\alpha \in \mathbb{T}$, the rotation of the cylinder is

$$R_\alpha(\theta, t) = (\theta + \alpha, t).$$

On the disk $\mathbb{D}$, R_α denotes rotation about the origin by angle $2\pi\alpha$. On the sphere $\mathbb{S}^2$, it denotes rotation by angle $2\pi\alpha$ around a fixed axis. If α is irrational, meaning that its class in $\mathbb{R}/\mathbb{Z}$ is not represented by a rational number, then R_α has no periodic points on the cylinder, exactly one periodic point on the disk, namely the center, and exactly two periodic points on the sphere, namely the two poles on the rotation axis. However, these rotations are not transitive on the whole surface: their orbits remain on horizontal circles in the cylinder, on concentric circles in the disk, and on latitude circles in the sphere.

For the purposes of this discussion, a symplectomorphism with the same extreme scarcity of periodic points is called a *pseudo-rotation*: it has no periodic points on $\mathbb{A}$, exactly one periodic point on $\mathbb{D}$, and exactly two periodic points on $\mathbb{S}^2$. The term does not mean that the map is close to a rotation. It means only that, from the viewpoint of periodic points, the map resembles an irrational rotation.

In 1941, Birkhoff (Birkhoff 1941) conjectured that an analytic symplectomorphism of the sphere $\mathbb{S}^2$ with only two fixed points and no other periodic points must be topologically conjugate to a rotation. He also conjectured that an analytic symplectomorphism of the cylinder $\mathbb{A}$ with no periodic points must be topologically conjugate to a rotation. In other words, Birkhoff predicted a strong rigidity phenomenon: in the analytic area-preserving world, the periodic-point pattern of an irrational rotation should force the entire system to be a rotation in disguise.

This conjecture remained open for more than 80 years, until Berger's work (Berger 2022), first posted online in 2022, disproved it.

Theorem 1 *There exist analytic symplectomorphisms of $\mathbb{A}$ and of $\mathbb{S}^2$, with respectively zero and two periodic points, that are not topologically conjugate to rotations.*

Thus periodic-point data alone do not force analytic rigidity. Berger's later work (Berger 2024) gives an even stronger failure of rigidity on the sphere and the disk: not only can analytic pseudo-rotations fail to be conjugate to rotations, they can have a dense orbit. This is especially striking because dense orbits are impossible for ordinary rotations of the disk or sphere.

Theorem 2 *There exists a transitive analytic symplectomorphism of $\mathbb{S}^2$ with exactly two periodic points, and there exists a transitive analytic symplectomorphism of $\mathbb{D}$ with exactly one periodic point.*

The transitivity condition in Theorem 2 immediately rules out conjugacy to a rotation. Indeed, if $h \circ f = g \circ h$, then

$$h\big(\mathrm{Orb}_f^+(x)\big) = \mathrm{Orb}_g^+\big(h(x)\big),$$

and a homeomorphism sends dense sets to dense sets. Hence transitivity is preserved by topological conjugacy. Since irrational rotations of the disk and the sphere are not transitive, the maps in Theorem 2 cannot be rotations in disguise, even though they have exactly the same number of periodic points as irrational rotations.

References

Berger, Pierre. 2022. "Analytic Pseudo-Rotations." *arXiv Preprint arXiv:2210.03438*.

Berger, Pierre. 2024. "Analytic Pseudo-Rotations II: A Principle for Spheres, Disks and Annuli." *arXiv Preprint arXiv:2402.15303*.

Birkhoff, George D. 1941. "Some Unsolved Problems of Theoretical Dynamics." *Science* 94: 598–600. https://doi.org/10.1126/science.94.2452.598.