A proof of the Poincaré conjecture

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This post continues my series on favorite theorems of the 21st century. For an overview of the categories and earlier selections, see this post.

My choice for 2003 in Geometry and Topology is Grigori Perelman's proof of the Poincaré conjecture. Indeed, this is my favourite theorem of the twenty-first century across all areas of mathematics. It is a theorem with a rare combination of qualities: the statement is simple enough to explain in a few sentences, the problem stood at the centre of topology for nearly a hundred years, and the proof brought together geometry, analysis, and topology in a spectacular way. As of this writing, the Poincaré conjecture remains the only solved Millennium Prize Problem.

The Poincaré conjecture concerns the topology of three-dimensional spaces. To state it, we first recall the notion of a manifold. Intuitively, a k-manifold is a space which, near each of its points, looks like ordinary Euclidean space $\mathbb{R}^k$. Thus a curve is a one-dimensional manifold, a surface is a two-dimensional manifold, and a three-manifold is a space which is locally three-dimensional.

More formally, a *k-dimensional topological manifold without boundary*, or simply a *k-manifold*, is a second-countable Hausdorff topological space in which every point has an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^k$. A compact k-manifold without boundary is called *closed*. Topology studies manifolds up to homeomorphism, that is, up to continuous deformation with a continuous inverse.

The basic model of a closed three-manifold is the *3-sphere*

$$S^3 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Although S^3 is naturally described as a subset of four-dimensional Euclidean space, it is itself three-dimensional: near every point, it looks like $\mathbb{R}^3$.

The two-dimensional analogue is the ordinary sphere

$$S^2 = \{(x_0, x_1, x_2) \in \mathbb{R}^3 : x_0^2 + x_1^2 + x_2^2 = 1\}.$$

Closed surfaces are completely classified. The *connected sum* of two surfaces is obtained by removing a disk from each surface and gluing the two resulting boundary circles together. The classification theorem for closed surfaces says that every connected closed surface is homeomorphic to exactly one of the following: the 2-sphere, the connected sum of g tori for some $g \geq 1$, or the connected sum of k real projective planes for some $k \geq 1$. In particular, the only simply connected closed surface is the 2-sphere.

Here a space is called *simply connected* if it is path connected and every loop in it can be continuously contracted to a point while remaining inside the space. For the 2-sphere this is true: any loop drawn on the surface of a sphere can be shrunk to a point. For a torus, it is false: a loop going around the hole cannot be contracted without cutting the surface.

Henri Poincaré asked whether the same characterization holds in dimension three. In 1904, he conjectured that every simply connected closed three-manifold is homeomorphic to the 3-sphere (Poincaré 1904). This became known as the *Poincaré conjecture*, one of the central problems of twentieth-century topology. In 2000, the Clay Mathematics Institute included it among the seven Millennium Prize Problems, each carrying a one-million-dollar prize for a correct solution (Carlson et al. 2006).

Progress first came in dimensions other than three. In 1960, Stephen Smale proved the corresponding high-dimensional statement in dimensions at least five (Smale 1960). In 1982, Michael Freedman proved the four-dimensional case (Freedman 1982). The original three-dimensional problem, however, remained open.

A much broader vision was proposed by William Thurston. In 1982, he formulated the *geometrization conjecture*, which predicted that every closed three-manifold can be decomposed into pieces carrying one of eight canonical geometric structures (Thurston 1982). This conjecture would imply the Poincaré conjecture, and it promised a classification theory for three-manifolds analogous in spirit to the classification theorem for closed surfaces.

In 2002 and 2003, Grigori Perelman posted three papers online in which he proved Thurston's geometrization conjecture, and hence the Poincaré conjecture (Perelman 2002, 2003b, 2003a).

Theorem 1 (Perelman) *Every simply connected closed three-manifold is homeomorphic to the 3-sphere.*

His proof built on Richard Hamilton's Ricci flow, an evolution equation which deforms a Riemannian metric in a way analogous to the diffusion of heat. Perelman's decisive innovations were to understand the singularities that form under the flow and to introduce a surgery procedure which cuts out highly curved regions while preserving enough information to continue the flow.

The result is one of the landmark achievements of modern mathematics. It solved a century-old problem, completed Thurston's vision of three-dimensional geometrization, and showed that deep questions in topology could be resolved by powerful geometric and analytic methods.

References

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