# Parametrization of quintic rings

Bogdan Grechuk • 19 Jun 2026

This post continues my series on favorite theorems of the twenty-first century. For an overview of the categories and earlier selections, see this post.

My choice for 2003 in Algebra is Manjul Bhargava's complete parametrization of quintic rings. The article was received by the *Annals of Mathematics* in 2003 and appeared in final form in 2008 (Bhargava 2008).

Throughout this post, a ring means a commutative ring with identity. Such a ring $R$ has *rank* $n$ if its additive group is a free abelian group of rank $n$, or equivalently if $R \cong \mathbb{Z}^n$ as a $\mathbb{Z}$-module. Rings of ranks $2, 3, 4$, and $5$ are called quadratic, cubic, quartic, and quintic rings, respectively.

The only rank-one ring is $\mathbb{Z}$. For quadratic rings, the classification is especially simple: the discriminant gives a bijection between isomorphism classes of quadratic rings and the set

$$\mathcal{D} = \{D \in \mathbb{Z} : D \equiv 0 \text{ or } 1 \pmod 4\}.$$

Thus quadratic rings are parametrized by their possible discriminants. For cubic rings, the corresponding theorem is much deeper: Delone and Faddeev (Delone and Faddeev 1964) showed that cubic rings are parametrized by integer equivalence classes of binary cubic forms. In modern language, this is a bijection between cubic rings and suitable $\mathrm{GL}_2(\mathbb{Z})$-orbits on integral binary cubic forms.

Bhargava's work showed that this pattern continues further than one might expect. In the quartic case, he proved that quartic rings, together with their cubic resolvent rings, are parametrized by pairs of integral ternary quadratic forms (Bhargava 2004). Here an integral ternary quadratic form is an expression

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz, \qquad a, b, c, d, e, f \in \mathbb{Z}.$$

More precisely, Bhargava constructed a canonical bijection between the appropriate

$$\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$$

orbits on pairs of such forms and isomorphism classes of pairs $(R, S)$, where $R$ is a quartic ring and $S$ is a cubic resolvent ring of $R$. For maximal quartic rings, the cubic resolvent is unique up to isomorphism, so this gives a genuine parametrization of maximal quartic rings themselves.

The quintic case is the next and most striking step. The general quintic equation is not solvable by radicals, but Bhargava showed that the integral theory of quintic rings still has a complete and beautifully structured parametrization. The right objects are no longer single forms, or pairs of forms, but quadruples of quinary alternating 2-forms. Concretely, one may think of a quinary alternating 2-form as a $5 \times 5$ skew-symmetric integer matrix. A quadruple of such forms is therefore an element of

$$\mathbb{Z}^4 \otimes \bigwedge^2 \mathbb{Z}^5,$$

a 40-dimensional lattice. The group

$$\mathrm{GL}_4(\mathbb{Z}) \times \mathrm{SL}_5(\mathbb{Z})$$

acts naturally on this lattice: $\mathrm{GL}_4(\mathbb{Z})$ changes the basis of the four forms, while $\mathrm{SL}_5(\mathbb{Z})$ changes the underlying five variables.

**Theorem 1** *There is a canonical bijection between the*

$$\mathrm{GL}_4(\mathbb{Z}) \times \mathrm{SL}_5(\mathbb{Z})$$

*orbits on*

$$\mathbb{Z}^4 \otimes \bigwedge^2 \mathbb{Z}^5$$

*and the isomorphism classes of pairs $(R, S)$, where $R$ is a quintic ring and $S$ is a sextic resolvent ring of $R$. Moreover, every quintic ring arises in this way, and if $R$ is maximal, then the corresponding orbit is unique.*

In other words, every quintic ring can be encoded by four $5 \times 5$ skew-symmetric integer matrices, with two encodings considered the same exactly when they differ by the natural change-of-basis operations above. The sextic resolvent ring plays the same role here that quadratic and cubic resolvents play in the cubic and quartic stories: it is the extra structure that makes the orbit statement clean and canonical. The explicit multiplication formulas are too long to reproduce here, but the shape of the answer is remarkably compact.

What I find most beautiful about the theorem is that it separates two very different meanings of “solving the quintic.” Abel’s theorem says that the general quintic equation cannot be solved by radicals. Bhargava’s theorem says that quintic rings nevertheless admit a complete arithmetic parametrization, just as quadratic, cubic, and quartic rings do. The classical formula disappears, but the hidden structure remains.

# References

Bhargava, Manjul. 2004. “Higher Composition Laws III: The Parametrization of Quartic Rings.” *Ann. of Math.* 159 (3): 1329–60.

Bhargava, Manjul. 2008. “Higher Composition Laws IV: The Parametrization of Quintic Rings.” *Ann. of Math.* 167 (1): 53–94.

Delone, Boris Nikolaevich, and Dmitrii Faddeev. 1964. *The theory of irrationalities of the third degree*. Vol. 10. American Mathematical Society.