Nash-Williams’ triangle decomposition conjecture is true

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In a recent preprint (Delcourt and Postle 2026), Delcourt and Postle proved the celebrated Nash-Williams triangle decomposition conjecture, a central open problem in extremal design theory. Their result confirms that there is a constant n_0 such that every graph G on $n \geq n_0$ vertices whose number of edges is divisible by 3, whose vertex degrees are all even, and whose minimum degree is at least $0.75n$, admits a decomposition of its edge set into edge-disjoint triangles.

We say that a graph G is *decomposed* into subgraphs H_1 and H_2, and write

$$G = H_1 \oplus H_2,$$

if G is the edge-disjoint, though not necessarily vertex-disjoint, union of H_1 and H_2. More generally, G is H*-decomposable* if

$$G = H_1 \oplus H_2 \oplus \cdots \oplus H_k,$$

where each H_i is isomorphic to H. The theory of graph decompositions asks under what conditions a graph G admits such a decomposition.

An obvious necessary condition for H-decomposability is H*-divisibility*. We say that a graph G is H-divisible if $|E(H)|$ divides $|E(G)|$ and if the greatest common divisor of the degrees of the vertices of H, denoted by $\gcd(H)$, divides the degree of every vertex of G. A central question in graph decomposition theory is when this necessary condition is also sufficient.

In a previous blog post, we discussed a theorem of Šajna (Šajna 2002), which states that if $H = C_m$ is a cycle of length m and the complete graph K_n is C_m-divisible, then K_n is C_m-decomposable. A far-reaching generalization of this theme concerns decompositions of non-complete graphs of large minimum degree into cycles of a fixed length. This is an active direction of research.

For each $m \geq 3$, the *cycle decomposition threshold* δ_m^C is defined as the infimum of all $\delta \geq 0$ such that every sufficiently large graph G on n vertices with minimum degree at least δn has a decomposition into cycles of length m if and only if m divides $|E(G)|$ and every vertex of G has even degree. In 2016, Barber, Kühn, Lo, and Osthus (Barber et al. 2016) reduced the problem in general to its natural fractional analogue. Using this reduction, they proved that

$$\delta_4^C = \frac{2}{3} \qquad \text{and} \qquad \delta_m^C = \frac{1}{2} \quad \text{for all even } m \geq 6.$$

For odd $m \geq 5$, the best known bounds are

$$\frac{1}{2} + \frac{1}{2m-2} \leq \delta_m^C \leq \frac{1}{2} + \frac{1}{2m-4},$$

see (Bryant et al. 2024).

The most famous case is $m = 3$. Nash-Williams (Nash-Williams 1970) observed that

$$\delta_3^C \geq \frac{3}{4}$$

and conjectured that equality holds. In other words, he conjectured that the obvious divisibility conditions for triangle decompositions become sufficient once the minimum degree is at least $0.75n$, for all sufficiently large graphs. In 2021, Delcourt and Postle (Delcourt and Postle 2021) proved the upper bound

$$\delta_3^C \leq \frac{7 + \sqrt{21}}{14} = 0.827\ldots.$$

In 2026, they posted a preprint (Delcourt and Postle 2026) proving the Nash-Williams conjecture in full.

Theorem 1 *For every $\epsilon > 0$ and every graph H with chromatic number 3, there is a constant $n_0 = n_0(H, \epsilon)$ such that every H-divisible graph G on $n \geq n_0$ vertices with minimum degree at least $(0.75 + \epsilon)n$ is H-decomposable. Moreover, when $H = K_3$, the same conclusion holds with $\epsilon = 0$.*

The authors first resolved the fractional version of the problem for $H = K_3$. Together with earlier work (Glock et al. 2019), this implies the first statement of Theorem 1. The “moreover” part, which gives the exact threshold for triangle decompositions, required substantially more work.

There is an analogous threshold problem for clique decompositions. Let δ_m^K denote the infimum of all $\delta \geq 0$ such that every sufficiently large graph G on n vertices with minimum degree at least δn has a decomposition into copies of K_m if and only if the necessary divisibility conditions hold; namely,

$$\binom{m}{2} \mid |E(G)| \qquad \text{and} \qquad m - 1 \mid d_G(v) \quad \text{for every } v \in V(G).$$

By definition, $\delta_3^K = \delta_3^C$, and so Theorem 1 proves that

$$\delta_3^K = \frac{3}{4}.$$

A folklore generalization predicted that

$$\delta_m^K = 1 - \frac{1}{m+1} \quad \text{for all } m \geq 3.$$

However, this was disproved in (Delcourt et al. 2025), where it was shown that, for some constant $c > 1$ and all $m \geq 4$,

$$\delta_m^K \geq 1 - \frac{1}{c(m+1)}.$$

References

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